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The physics and mathematics of Calogero particles

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Abstract

We give a review of the mathematical and physical properties of the celebrated family of Calogero-like models and related spin chains.

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Contents

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1. Introduction

The system of one-dimensional particles with inverse-square pairwise interactions has assumed a 'Jack-in-the-box' role in mathematical and theoretical physics over the last three decades: it pops up in various seemingly disparate situations, it is a recurring and unifying theme in high energy and condensed matter physics and constitutes the prime example of an integrable and solvable many-body system. Its range of applicability spans fluid mechanics, spin chains, gauge theory and string theory. This model has been labelled in the literature with various subsets and permutations of the last names of Francesco Calogero, Bill Sutherland and Jurg Moser. We shall simply call it the Calogero model, for brevity and in recognition of its original inventor.

The Calogero model (and its many generalizations) has reached maturity, in the sense that its various aspects (classical, quantum, differential equation, operator, statistical, symmetry, etc) have been exhaustively analysed and no new groundbreaking results seem to appear recently (I could be wrong!). It is unlikely, however, that it will be relegated to the shelves of mathematical physics for perpetuity. It has already enjoyed several revivals in its 35-odd-year history and there are still open issues awaiting resolution. Chances are there that it has a few more surprises up its sleeve, their unveiling requiring, and providing back, new intuition.

The purpose of this paper is to expose the interesting physical and mathematical properties of the system and whet the appetite of readers for further study and research on the topic. It is not meant to be exhaustive, or rigorous, or all-encompassing: there are excellent extensive review articles that can serve this purpose. The hope is that the small size, physical slant and stress on newer developments will make this paper an accessible and motivational first piece.

2. Basic features of the Calogero model

2.1. Introducing the model

The prototype of the model is the system of identical particles scattering on the line with inverse-square interaction potentials, as first introduced by Calogero [1]. Its Hamiltonian (in an obvious notation) is

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i<j} \frac{g}{(x_i - x_j)^2}. \tag{1}$$

The particle masses m have been scaled to unity.

The motivation for such a model comes from various perspectives. This is the only scale-free two-body potential that one can have: quantum mechanically, the potential scales like the kinetic term and therefore the coupling g becomes dimensionless (in $\hbar = 1$ units). The inverse-square potential would arise as a centrifugal term in higher dimensions and is 'borderline' from the sense of stability: anything stronger than that at short distances and the

particles would 'fall' into each other. Finally, from the long-distance point of view, the inverse-square potential is borderline for statistical mechanics phase transitions: a stronger potential leads to phase transitions, while a weaker one does not. The Calogero model 'straddles the line' in many respects.

To make the system bound, one has to introduce a kind of external 'box'. One way of doing this is to include an external harmonic oscillator potential that confines the system. This is nice, since it does not spoil the basic features of the system and leads to an integrable model, the harmonic Calogero model. Extracting thermodynamics from this system, however, is a bit tricky since the box is not homogeneous and the process requires careful scaling. Alternatively, one could put the system in a finite periodic box (whose length can be scaled to 2π). The particles now interact through all the infinitely many periodic images of themselves and the two-body potential becomes

$$V(x) = \sum_{n=-\infty}^{\infty} \frac{g}{(x+2\pi n)^2} = \frac{g}{\left(2\sin\frac{x}{2}\right)^2}. \tag{2}$$

This is the Sutherland model [2]. There are other versions of this class of models that we will not analyse. The original papers on the subject still consist essential reading [1–11]. The classic report [5] analyses these systems in detail, the paper [12] describes relativistic analogues of these systems (known as the Ruijsenaars–Schneider model) that are not touched in the present paper, the lectures [13] cover many issues related to fractional statistics and [14] provides further mathematical results.

2.2. *Stability, Hermiticity*

Classically, the coupling constant g should be positive to ensure that particles are not 'sucked' into each other. Quantum mechanically, the uncertainty principle works in our favour and the minimum allowed value for g is $g = -\frac{1}{4}$ (we put henceforth $\hbar = 1$). This result can be derived by regularizing the singularity of the potential at $x_i - x_j = 0$ and taking the limit, as done, e.g., in Landau and Lifshitz's classic quantum mechanics book.

The Hermiticity properties of the system for negative coupling constants have received a lot of attention and there are ways to make sense of even values less than $g = -\frac{1}{4}$. It is not our purpose to analyse these here, since it seems that the applications that arise naturally are those respecting the above condition.

For later convenience, it is useful to parametrize g in the fashion

$$g = \ell(\ell - 1), \tag{3}$$

in which case the minimum value is naturally obtained for $\ell = \frac{1}{2}$. Note, however, that there are generically *two* values of ℓ that give the same g, namely ℓ and $\ell' = 1 - \ell$. For $-\frac{1}{4} < g < 0$, in particular, they are both positive. It is possible to argue that only the value $\ell \geqslant \frac{1}{2}$ is relevant, although there are good reasons to retain both of them for the case of negative g. The possible use of that will become clear in the following.

2.3. *Properties of the classical motion*

The above system is classically integrable, which means that there are N integrals of motion in convolution, that is, N functions on phase space with vanishing Poisson brackets:

$$\{I_n, I_m\} = 0, \qquad n, m = 1, \ldots, N. \tag{4}$$

For the scattering system (no external potential), I_1 is the total momentum, I_2 is the total energy and higher I_n are higher polynomials in the momenta also involving the two-body potentials.

The integrals of motion are derived from the so-called Lax matrix of the model. Consider the $N \times N$ Hermitian matrices

$$L_{jk} = p_j \delta_{jk} + (1 - \delta_{jk}) \frac{\mathrm{i}\ell}{x_{jk}}, \qquad M_{jk} = \ell \delta_{jk} \sum_{s=1}^{N} \frac{1}{x_{js}^2} + \ell(\delta_{jk} - 1) \frac{1}{x_{jk}^2}, \tag{5}$$

with $\ell^2 = g$ and $x_{jk} = x_j - x_k$. Then, upon use of the equations of motion of the Calogero model, the evolution of the elements of the matrix L is

$$\dot{L} = \mathrm{i}[L, M]. \tag{6}$$

A pair of matrices satisfying this equation is called a Lax pair, with L the Lax matrix. What the above equation means is that the evolution of L is simply a unitary conjugation $L \to ULU^{-1}$ generated by the (time-dependent) matrix M. So all the eigenvalues of L are conserved and the traces

$$I_n = \operatorname{tr} L^n, \qquad n = 1, \ldots, N, \tag{7}$$

are constants of motion. Obviously, $I_1 = \sum_i p_i$, while explicit calculation and the use of the identity

$$\sum_{i,j,k \text{ distinct}} \frac{1}{x_{ij} x_{ik}} = 0 \tag{8}$$

(easily proved by multiplying numerator and denominator by x_{jk} and cyclically redefining the dummy indices i, j, k) shows that $I_2 = 2H$. I_3 and beyond generate new, nontrivial conserved quantities.

It can further be shown that the above integrals are in involution, that is, they have vanishing Poisson brackets. For the scattering system, it is quite easy to give a physical proof: at $t \to \infty$, the particles fly far away from each other and L becomes $\operatorname{diag}(p_i)$, the off-diagonal elements asymptotically vanishing. The integrals of motion simply become

$$I_n = \sum_i k_i^n, \tag{9}$$

with k_i the asymptotic momenta. Obviously, these are in involution. Since the Poisson brackets of conserved quantities are also conserved, they must vanish at all times.

The key interesting property of the above model, which sets it apart from other merely integrable models, is that, both classically and quantum mechanically, it mimics as closely as possible a system of free particles. We shall give here an overview of these properties, without insisting on proofs, and will come back to their derivation in subsequent sections.

Let us first look at its classical behaviour. The motion is a scattering event. Asymptotically, at times $t = \pm\infty$, the particles are far away, the potentials drop off to zero and motion is free. When they come together, of course, they interact and steer away from their straight paths. Interestingly, however, when they are interacting, they resume their previous free paths as if nothing happened. Not only are their asymptotic momenta the same as before scattering, but also the asymptotic positions (scattering parameters) are the same. There is no time delay of the particles in the scattering region. The only effect is an overall reshuffling of the particles. Thus, if one cannot distinguish particles from each other, and if one only looks at scattering properties, the system looks free!

Similar behaviour is exhibited by the harmonic Calogero and the Sutherland model. The motion of the harmonic model is periodic, with period determined by the harmonic oscillator

potential. The particles seem to revolve around ghostly paths of would-be non-interacting particles in the external potential, their interaction creating deviations but never completely upsetting the system. For the Sutherland model, the particles seem bound to a set of free particle paths corresponding to a given set of momenta, deviating from them as they come near each other and resuming them as they separate. The momenta of the free paths (called 'pseudomomenta') are related to the integrals of motion of the system.

Is there a concise manifestation of the coupling constant g in the classical motion of the particle, except its details? It turns out that there is a particularly neat result involving g: the classical action of two particles scattering off each other is the same as the action of free particles with the same asymptotic momenta, but diminished by

$$\Delta S = \pi\sqrt{g}. \tag{10}$$

Semiclassically, the action corresponds to the phase of the wavefunction for the scattering process. This suggests that the scattering phase shift differs by a non-dynamical constant from the corresponding free particle result. As we shall see, this is fully born out in quantum mechanics.

2.4. Properties of the quantum system: fractional statistics

The above behaviour carries over to quantum mechanics. The asymptotic scattering momenta are the same before and after scattering. The classical fact that there is no time delay translates into the quantum fact that the scattering phase shift is independent of the momenta (remember that the time delay is the momentum derivative of the phase shift). Thus, it can only be a function of the coupling constant and the total number of particles. It is, actually, a very suggestive function:

$$\theta_{\rm sc} = \frac{N(N-1)}{2}\ell\pi. \tag{11}$$

So the phase is simply $\ell\pi$ times the total number of two-body exchanges that would occur in the scattering of free particles. This squares with the previous classical result: the classical limit corresponds to $\ell \gg 1$, so $\sqrt{g} \simeq \ell$ and we get $\Delta S = \theta_{\rm sc}$.

We can interpret the above property as the fact that Calogero particles are essentially free but obey generalized statistics, as defined by their scattering phase shift [7]. Clearly, the case $\ell = 0$ would correspond to free bosons and $\ell = 1$ to free fermions (for these two values the potential vanishes and the system is, indeed, free). For any other value we can say that the system has fractional statistics of order ℓ. Note that, in contrast to anyons, ℓ is *not* a phase: it can be fractional but also bigger than 1 ('superfermions'?).

A word on the permutation properties of this system is in order. The inverse-square potential is quantum mechanically impenetrable, and thus the 'ordinary' statistics of the particles (symmetry of the wavefunction) is irrelevant: if the particle coordinates are in one of the $N!$ ordering sectors, they will stay there for ever. The wavefunction could be extended to the other sectors in a symmetric, antisymmetric or any other way, but this is irrelevant for physics. No interference between the sectors will ever take place. All states have a trivial $N!$ degeneracy. (That is to say, all states in all irreps of S_N have the same physical properties.) Permutation statistics are therefore irrelevant and we can safely talk about the effective statistics as produced by their coupling constant ℓ.

Let us also review the properties of the confined systems. In the presence of an external harmonic potential of the form

$$V = \sum_i \frac{1}{2}\omega^2 x_i^2, \tag{12}$$

the energy spectrum of a system of uncoupled particles would be

$$E = \frac{N}{2}\omega + \sum_i n_i \omega. \tag{13}$$

n_i are nonnegative integers satisfying

$$n_1 \leqslant \cdots \leqslant n_N \qquad \text{for bosons,} \tag{14}$$

$$n_1 < \cdots < n_N \qquad \text{for fermions.} \tag{15}$$

The actual spectrum of this model is

$$E = \frac{N}{2}\omega + \ell\frac{N(N-1)}{2}\omega + \sum_i n_i \omega, \tag{16}$$

with n_i being 'excitation numbers' obeying bosonic selection rules: $n_i \leqslant n_{i+1}$. Defining the 'pseudoexcitation numbers'

$$\bar{n}_i = n_i + (i-1)\ell, \tag{17}$$

we can check that the expression of the spectrum in terms of $\bar{n}_i$ is identical to the free one (13) but with the quantum numbers now obeying the selection rule

$$\bar{n}_i \leqslant \bar{n}_{i+1} - \ell. \tag{18}$$

This is a sort of exclusion principle that requires the particle quantum numbers to be at least a distance ℓ apart (as contrasted to 1 for fermions and 0 for bosons). Again, a generalized statistics interpretation is manifest [17].

Let us clarify that the above numbers $\bar{n}_i$ are no more integers. They do, however, increase in integer increments. The rule is that the ground state is determined by the minimal allowed nonnegative values for $\bar{n}_i$ obeying (18) while the excited states are obtained by all integer increments of these values that still obey (18).

The periodic (Sutherland) model has similar properties. Its spectrum is

$$E = \sum_i \frac{1}{2}k_i^2 + \ell\sum_{i<j}(k_j - k_i) + \ell^2\frac{N(N^2-1)}{24}, \tag{19}$$

with the 'momenta' k_i being integers satisfying bosonic rules: $k_i \leqslant k_{i+1}$. This looks rather different than the corresponding free expression (for $\ell = 0$). Defining, however, again 'pseudomomenta'

$$p_i = k_i + \ell\left(i - \frac{N+1}{2}\right), \tag{20}$$

we can check that the expression for the spectrum becomes

$$E = \sum_i \frac{1}{2}p_i^2, \tag{21}$$

that is, the free expression. The pseudomomenta satisfy

$$p_i \leqslant p_{i+1} - \ell, \tag{22}$$

that is, the same selection rule as $\bar{n}_i$ before. Again, we observe a generalization of the fermionic and bosonic selection rules corresponding to statistics ℓ. The ground state is the minimal (nearest to zero) numbers satisfying (22) while excitations correspond to integer increments (or decrements) of p_i still satisfying (22).

Let us note that the above rule for $\ell = 1$ reproduces the fermionic spectrum of particles with periodic boundary conditions for odd N and *anti*-periodic ones for even N: in the odd (even) N case, the momenta are quantized to (half-) integers. This has a natural interpretation: when we take a particle around the circle it goes over $N - 1$ other particles. If we require the phase shift of the wavefunction in this process to agree with the minus signs picked up from the $N - 1$ fermion exchanges, we recover the previous rule. We stress that, for free particles, this is not a consistency requirement but rather an aesthetic rule. At any rate, this is what the Sutherland model chooses to do!

In conclusion, we see that the Calogero model can be though of as a system of particles obeying generalized statistics. This manifests in terms of the scattering phases and, most significantly for statistical mechanics, through a peculiar 'level repulsion' of their quantum numbers generalizing the Fermi exclusion principle.

2.5. Large-N properties of the Calogero model and duality

Let us examine, now, the properties of the Calogero model as the number of particles grows large. At zero temperature, a non-interacting fermion system would form a Fermi sea. The corresponding 'Fermi surface' in one dimension degenerates to points. For the system in an external harmonic potential, there is just one point corresponding to the highest excitation $n_{\mathrm{F}} = N-1$. For the free periodic system, we would have two Fermi momenta at $\pm p_{\mathrm{F}} = \pm\frac{N-1}{2}$. Excitations over this ground state are, then, conveniently classified in terms of particles (a filled Fermi sea with an isolated particle above or below) and holes (a filled sea with one unoccupied state inside it).

Interestingly, the Calogero model presents a similar picture. The qualitative features of both the Calogero and the periodic Sutherland model are similar, so we pick the latter as most closely representing a gas of free particles in a box. From (22) we see that the ground state also forms a 'pseudo-Fermi sea' (or should we call it a 'Luttinger sea'?) with Fermi levels rescaled by a factor ℓ: $p_{\mathrm{F}} = \ell\frac{N-1}{2}$. Its minimal excitations are analogous to those of a Fermi sea, but not quite:

- A particle would be an isolated occupied pseudomomentum above or below a completely filled sea of pseudomomenta. Particles
  - are excited in units of 1 (the increments of their pseudomomentum).
  - take up a space ℓ in pseudomomentum (since they cannot be 'packed' closer than ℓ units apart).
- A hole would be an isolated empty space inside an otherwise occupied sea. Interestingly, the minimal such excitation is *not* obtained by removing one particle from the sea, but rather by incrementing all pseudomomenta of the sea above the place where we want to create the hole by one unit. Holes
  - are excited in units of ℓ. Indeed, since the distance of pseudomomenta in the sea is ℓ, the possible positions of the hole are at distances ℓ apart.
  - take up a unit space in pseudomomentum. Indeed, incrementing all pseudomomenta above a given place in the sea by *two* units creates two holes in that place, and so on; by locally reshuffling pseudomomenta we can then separate these holes.

Note that holes are *not* antiparticles. Removing a particle for the sea creates a gap of ℓ spaces and, from above, ℓ holes. The correspondence is

$$1 \text{ particle} \sim -\ell \text{ holes}. \tag{23}$$

We already observe a sort of duality between the two types of excitations. This can be summarized as

$$\text{particle} \leftrightarrow \text{hole}, \qquad \ell \leftrightarrow \frac{1}{\ell}, \qquad p \leftrightarrow \ell p. \tag{24}$$

Under the above, the spectrum of excitations of the model remains invariant. This is the simplest manifestation of a coupling-constant duality that goes over to the correlation functions and Green's functions of the model [18–21]. Obviously, this duality is spoiled by nonperturbative effects, since holes are confined within the sea while there is no 'ceiling' for particle excitations.

3. Particle symmetries and the Calogero model

So far we have talked about the Calogero model and its properties considering it a 'given' system, with only circumstantial motivation. It is now time to 'derive' the model, in the sense of obtaining it by starting from a set of principles.

The angle we will take is that of the description of a set of indistinguishable particles and their symmetries. Field theorists are used to the idea of viewing particles as field quanta or as representations of the Poincaré group. The many-body (first quantized) point of view, however, affords a perspective revealing the Calogero model as the natural generalization of free indistinguishable particles.

The main idea is that *identical* particles admit the permutation group as a dynamical symmetry commuting with the Hamiltonian, while *indistinguishable* particles elevate the permutation group to a (discrete) *gauge symmetry*. In plain words, configurations in which particles are permuted correspond to the same physical state and constitute 'gauge' copies of the system.

There are two distinct ways to deal with a gauge system:

- Reduce the system to a set of gauge-invariant observables.
- Realize the gauge symmetry as a symmetry of the Hilbert space and impose gauge constraints on the states.

We shall explain below how each of these will lead to (versions of) the Calogero model.

3.1. Reducing to gauge-invariant observables

In the sense exposed above, phase space particle coordinates x_i, p_i are not physical observables since they insist on assigning a label to each particle; they are not permutation invariant. A set of invariants can be constructed in terms of symmetric functions of the above coordinates. Such a set is

$$I_{n,m} = \sum_{i=1}^{N} : x_i^n p_i^m : \tag{25}$$

with $n, m \geqslant 0$ and $: \cdot :$ denoting a specific ordering. For example, the symmetric (Weyl) ordering between x_i and p_i can be adopted, which also ensures the Hermiticity of $I_{m,n}$.

The above invariants are, in general, overcomplete. Even if we were to truncate the range of n, m to N, there would still be of the order of N^2 observables, while the number of independent basic operators is $2N$. Classically, this redundancy manifests in the existence of algebraic identities between various $I_{m,n}$.

Quantum mechanically, overcompleteness translates into the presence of Casimirs in the algebra of $I_{m,n}$. Indeed, $I_{m,n}$ satisfy (a particular parametrization of) the so-called W_N algebra.

In the $N \to \infty$ limit, their commutation relations can be conveniently repackaged into the 'sine algebra' [15], by defining

$$I(k,q) = \sum_{m,n=0}^{\infty} \frac{k^m q^n}{m!n!} I_{m,n}, \tag{26}$$

with k, q continuous 'Fourier' variables. Then, assuming Weyl ordering for $I_{m,n}$, $I(k,q)$ satisfy

$$[I(k,q), I(k',q')] = 2\mathrm{i} \sin \frac{kq' - k'q}{2} I(k+k', q+q'), \tag{27}$$

from which the commutators of $I_{m,n}$ can be obtained by Taylor expanding $I(k,q)$ and matching coefficients of $k^m q^n$. To lowest order in $\hbar$, we obtain

$$[I_{m,n}, I_{m'n'}] = \mathrm{i}(mn' - nm') I_{m+m'-1,n+n'-1} + O(\hbar^2), \tag{28}$$

which is the 'classical' W_∞ algebra. (We have put $\hbar = 1$, but lowest order in $\hbar$ corresponds to lowest order in m, n.)

For finite N, the algebra becomes nonlinear due to the presence of identities. Alternatively, we can keep the full tower of $I_{m,n}$ and effectively impose the identities as relations for the Casimirs of the algebra. Indeed, it is known that the above algebra admits a host of representations, corresponding to the underlying particles being bosons, fermions, parabosons or parafermions and various other possibilities.

To see how the Calogero system emerges as one of these possibilities, concentrate, for the moment, on the special case of two particles [16]. Their centre-of-mass coordinate and momentum

$$X = \frac{x_1 + x_2}{2}, \qquad P = p_1 + p_2 \tag{29}$$

are certainly gauge-invariant observables; they correspond to $I_{1,0}$ and $I_{0,1}$. The relative coordinate and momentum, however,

$$x = x_1 - x_2, \qquad p = \frac{p_1 - p_2}{2} \tag{30}$$

are not, since they are odd under permutation. We can form quadratic invariants as

$$A = x^2, \qquad B = \frac{xp + px}{2}, \qquad C = p^2. \tag{31}$$

They correspond to

$$A = 2I_{2,0} - I_{1,0}^2, \qquad B = I_{1,1} - \tfrac{1}{4}(I_{1,0}I_{0,1} + I_{0,1}I_{1,0}), \qquad C = \tfrac{1}{2}I_{0,2} - \tfrac{1}{4}I_{0,1}^2. \tag{32}$$

The relative variables A, B, C commute with the centre-of-mass variables and close to the $SL(2, R)$ algebra

$$[A, B] = \mathrm{i}2A, \qquad [B, C] = \mathrm{i}2B, \qquad [A, C] = \mathrm{i}4B. \tag{33}$$

Classically, the above variables satisfy the constraint $AC = B^2$. Quantum mechanically, this translates to the Casimir

$$G = \frac{AC + CA}{2} - B^2. \tag{34}$$

Physical Hilbert spaces correspond to irreducible representations of the algebra of A, B, C, along with X, P.

The representation with vanishing Casimir corresponds to the original system of two bosons or two fermions, in which A, B, C can be realized as in (31). This is not, however, the only one. A nonzero value for G, corresponding to a quantum correction to the classical value,

would also be a legitimate realization of the indistinguishable particle dynamics. Interestingly, unitarity mandates that $G \geqslant -\frac{1}{2}$, so we may parametrize

$$G = \ell(\ell - 1) \tag{35}$$

in direct analogy to the Calogero case. We observe that the representation with the above value of G can be realized as

$$A = x^2, \qquad B = \frac{xp + px}{2}, \qquad C = p^2 + \frac{\ell(\ell - 1)}{x^2}. \tag{36}$$

Effectively, the relative kinetic energy of the particles has acquired an inverse-square potential part. The free particle Hamiltonian for this system would become

$$H = \frac{P^2}{2} = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{\ell(\ell - 1)}{(x_1 - x_2)^2}. \tag{37}$$

This is the Calogero model! The Calogero coupling plays the role of the Casimir. The point is that the Hamiltonian and other observables of the Calogero model and the corresponding 'free' system (free of inverse-square interactions) in terms of physical variables X, P, A, B, C are identical.

For more than one particle, the approach can be extended and we again recover the Calogero model as one particular realization of the indistinguishable particle system.

The Hilbert space of the above realization, in terms of x_i, p_i, still provides a defining $N!$-dimensional realization of the permutation algebra. For generic values of the Calogero coupling, however, this is inconsequential: the inverse-square potential is impenetrable (both classically and quantum mechanically) and thus the $N!$ sectors do not mix and are physically equivalent. There is no need to decompose the defining representation into irreducible components. The 'normal' statistics of the particles (wavefunction symmetry properties) have become immaterial, being totally supplanted by the Calogero dynamics. Only for $G = 0$, that is, $\ell = 0$ or 1, the Calogero interaction vanishes and ordinary statistics come back into play. By analytic continuation of the wavefunction properties, we can map bosons to the $\ell = 0$ case and fermions to the $\ell = 1$ case.

3.2. Augmenting the symmetry: matrix model

The opposite way to realizing a gauge system is to keep the original redundant formulation and impose gauge invariance as an operator relation on the states (Gauss' law). In terms of the single-particle phase space variables x_i, p_i, the gauge symmetry is the permutation (symmetric) group $S = N$. This leads to the well-known and analysed cases of fermions, bosons and their parastatistics generalizations.

A different approach is to start with an *augmented* system, in which both the dynamical variables and the gauge symmetry have been expanded, giving at the end the same gauge-invariant degrees of freedom [22]. Specifically, we could formulate the particles in terms of the eigenvalues of an $N \times N$ matrix. There is no *a priori* ordering of these eigenvalues, so this certainly encodes identical particles. It is clear that the permutation symmetry of the problem has been promoted to the continuous symmetry of unitary conjugations of this matrix, which leaves the eigenvalues intact.

This unitary conjugation symmetry is now the gauge group. States need not necessarily be singlets under this symmetry, however, just as in the previous section the Casimir G need not vanish. We can simply select the Hilbert space to transform under an irreducible representation of the gauge group and identify states transforming to each other under the action of the group as a unique physical state, in direct analogy to the considerations that lead to parastatistics.

This is the celebrated matrix model formulation, to be fully analysed in the subsequent sections. It has found various applications in physics, the most directly related to identical particles, perhaps, being the noncommutative Chern–Simons description of the finite quantum Hall droplet [23–27].

4. The Hermitian matrix model

4.1. Classical analysis

Let us first examine a matrix model that parallels as closely as possible particle mechanics on the infinite line. The kinematical variable is a Hermitian $N \times N$ matrix M and the Lagrangian reads

$$\mathcal{L} = \mathrm{tr}\left\{\tfrac{1}{2}\dot{M}^2 - V(M)\right\}. \tag{38}$$

$V(x)$ is a scalar potential evaluated for the matrix variable M.

Clearly, the above has a time-translation invariance which leads to the conserved energy

$$H = \mathrm{tr}\left\{\tfrac{1}{2}\dot{M}^2 + V(M)\right\}. \tag{39}$$

Moreover, the action is invariant under time-independent unitary conjugations of the matrix M:

$$M \to UMU^{-1}. \tag{40}$$

This non-Abelian $SU(N)$ symmetry leads to the conserved Hermitian traceless matrix

$$J = \mathrm{i}[M, \dot{M}], \tag{41}$$

where $[\cdot, \cdot]$ denotes ordinary matrix commutator. These are the 'gauge charges' that, when fixed, will determine the particular realization ('statistics') of the indistinguishable particle system. But let us further analyse the implications of fixing these charges classically.

We are interested in the dynamics of the eigenvalue of M, so we parametrize it as

$$M = U\Lambda U^{-1}, \tag{42}$$

where $U(t)$ is the unitary 'angular' part of the matrix and $\Lambda(t) = \mathrm{diag}\{x_1, \ldots, x_N\}$ are the eigenvalues. Clearly, the conserved quantity J has to do with invariance under 'rotations' of the angular part of M and thus corresponds to the 'angular momentum' of $U(t)$. We define the 'gauge potential'

$$A = -U^{-1}\dot{U}. \tag{43}$$

$\dot{M}$, J and the Lagrangian $\mathcal{L}$ become, in this parametrization,

$$\dot{M} = U(\dot{\Lambda} + [\Lambda, A])U^{-1}, \tag{44}$$

$$J = \mathrm{i}U([\Lambda, [\Lambda, A]])U^{-1} \equiv UKU^{-1}, \tag{45}$$

$$\mathcal{L} = \mathrm{tr}\left\{\frac{1}{2}\dot{\Lambda}^2 + [\Lambda, A]^2 - V(\Lambda)\right\} \tag{46}$$

$$= \frac{1}{2}\sum_{i=1}^{N}\dot{x}_i - \frac{1}{2}\sum_{i,j=1}^{N}(x_i - x_j)^2 A_{ij}A_{ji}. \tag{47}$$

The matrix elements of A and K are related as

$$K_{jk} = \mathrm{i}[\Lambda, [\Lambda, A]]_{jk} = \mathrm{i}(x_j - x_k)^2 A_{jk}. \tag{48}$$

Finally, solving (48) for A_{jk} and putting into (47) we obtain

$$\mathcal{L} = \sum_i \frac{1}{2}\dot{x}_i^2 + \frac{1}{2}\sum_{i\neq j} \frac{K_{ij}K_{ji}}{(x_i - x_j)^2} - \sum_i V(x_i). \tag{49}$$

The first two terms are kinetic, coming from $\dot{M}^2$, while the last one is potential. Therefore, the Hamiltonian H is

$$H = \sum_i \frac{1}{2}p_i^2 + \frac{1}{2}\sum_{i\neq j} \frac{K_{ij}K_{ji}}{(x_i - x_j)^2} + \sum_i V(x_i). \tag{50}$$

Note that the eigenvalues are kinematically coupled by an inverse-square type potential with the angular momentum degrees of freedom. The connection of the matrix model to the Calogero model along the lines presented here and below was first established in [4]. Also, the Hamiltonian (50) has been proposed independently of the matrix model as an $SU(N)$-generalization of the classical Calogero system [6].

We can now examine special cases:

(a) The most 'gauge-invariant' sector is, of course, that in which the angular momentum charges vanish, that is, $J = 0$. In that case, (50) for $K = 0$ becomes the Hamiltonian of non-interacting particles in an external potential $V(x)$. This would be the case of 'standard' particles.

(b) For the next simplest case, the angular momentum J should be as simple as possible without vanishing. Only its eigenvalues are really relevant, since we can always perform a time-independent unitary transformation V which would shift $U \to VU$ and would rotate $J \to VJV^{-1}$. The simplest choice would be to take the eigenvalues of J to be equal. Unfortunately, this is not possible since the traceless condition would make them vanish. The simplest possible choice is to take all the eigenvalues equal to ℓ except one, which would cancel the trace by being $(1 - N)\ell$. This can be written in terms of an arbitrary column N-vector v as

$$J = \ell(vv^\dagger - 1), \qquad v^\dagger v = N, \tag{51}$$

in which case K becomes

$$K = \ell(uu^\dagger - 1), \qquad u = U^{-1}v. \tag{52}$$

From (48) we see that $K_{ii} = 0$ (no sum on i) and thus

$$u_i u_i^* = 1 \qquad \text{(no sum)}. \tag{53}$$

So the coefficient of the inverse-square potential in (50) becomes

$$K_{ij}K_{ji} = \ell u_i u_j^* \, \ell u_j u_i^* = \ell^2 \qquad (i \neq j). \tag{54}$$

Finally, (50) becomes

$$H = \sum_i \frac{1}{2}p_i + \sum_{i<j} \frac{\ell^2}{(x_i - x_j)^2} + \sum_i V(x_i). \tag{55}$$

This is the Calogero model! The potential strength $g = \ell^2$ is related to the conserved charge ℓ. Quantum mechanically, picking this charge will amount to a choice of statistics. We also get, at this stage, an arbitrary external potential $V(x)$.

(c) More general choices of J amount to more variety in its eigenvalues. $K_{ij}K_{ji}$ now, in general, becomes time dependent and the dynamics more complicated. We postpone the discussion for the quantum case where it will be shown that this corresponds to Calogero particles having also internal degrees of freedom. This will be a generalization of the discussion of the first section, with irreps of $SU(N)$ substituting the irreps of S_N.

Now that we have this new approach we can use matrix technology to demonstrate the integrability of the Calogero model [4, 5]. For $V(x) = 0$, the matrix motion becomes free and $\dot{M}$ is conserved. The conjugation-invariant quantities

$$I_n = \operatorname{tr} \dot{M}^n \tag{56}$$

are also conserved and are in involution (the matrix elements of $\dot{M}$ are momenta and have vanishing Poisson brackets). From (44) and (48) we have

$$\dot{M}_{jk} = U^{-1} \left(\delta_{jk} \dot{x}_j - (1 - \delta_{jk}) \frac{\mathrm{i} K_{jk}}{x_j - x_k} \right) U \tag{57}$$

$$= U^{-1} \left(\delta_{jk} \dot{x}_j - (1 - \delta_{jk}) \frac{\mathrm{i} u_j u_k^*}{x_j - x_k} \right) U. \tag{58}$$

When the above expression is inserted in the trace $I_n = \operatorname{tr} \dot{M}^n$, clearly U drops and products of the form $u_i u_j^* u_j u_k^* \cdots u_i^*$ will appear which reduce to powers of ℓ. Therefore, I_n reduce to expressions involving only $x_i, \dot{x}_i$ and the coupling constant ℓ. These are the conserved integrals of the Calogero model.

Starting from the matrix model, the actual motion of the Calogero model can be obtained. The solution for M is

$$M = B + Ct \tag{59}$$

for arbitrary matrices B, C. The conserved charge becomes

$$J = \mathrm{i}[M, \dot{M}] = \mathrm{i}[B, C] = \mathrm{i}\ell(uu^\dagger - 1). \tag{60}$$

By unitary transformations we can choose the phases of u such that $u_i = 1$; choices for B, C, then, that satisfy (60) are

$$B_{jk} = \delta_{jk} q_j, \qquad C_{jk} = \delta_{jk} p_j - (1 - \delta_{jk}) \frac{\mathrm{i}\ell}{q_j - q_k}. \tag{61}$$

q_i and p_i are the initial conditions for x_i and $\dot{x}_i$ at time $t = 0$. Diagonalizing, then, $M = B + Ct$ for the above B, C produces the motion of the system. Another choice is

$$B_{jk} = \delta_{jk} a_j + (1 - \delta_{jk}) \frac{\mathrm{i}\ell}{P_j - P_k}, \qquad C_{jk} = \delta_{jk} P_j. \tag{62}$$

P_i and a_i are asymptotic momenta and impact parameters. For $t \to \pm\infty$, the off-diagonal elements of B produce a perturbation of order t^{-1} to the eigenvalues, so the motion is determined by the diagonal elements $a_i + P_i t$ alone. We recover the result that the motion in asymptotic regions is the same as if the particles were free.

We conclude by proving that the matrix model is also integrable and solvable in the presence of a harmonic oscillator potential $V(x) = \frac{1}{2}\omega^2 x^2$. The non-Hermitian matrix $Q = \dot{M} + \mathrm{i}\omega M$ evolves as

$$Q(t) = \mathrm{e}^{\mathrm{i}\omega t} Q(0) \tag{63}$$

and the matrix $Q^\dagger Q$ is conserved. We leave it as an exercise to derive the conserved integrals and the motion of the corresponding Calogero problem.

External potentials with up to quartic dependence on x also lead to integrable, although not so solvable, models [28]. It is an open question to prove that this is all there is or to find yet more integrable potentials.

Finally, we may wonder what restricts us to one dimension. We chose a model with one matrix, and its eigenvalues corresponded to coordinates of particles on the line. We could,

indeed, start with an appropriate model with many matrices, which would reproduce particle motion in higher dimensions [29]. The integrability and solvability properties of such extended models, however, are much less pleasant. The question of whether they represent a workable extension of identical particles remains open.

4.2. The Hermitian matrix model: quantum

We will, now, perform the quantization of this system. We will do it first for the Hermitian matrix model and subsequently for the unitary one. Each model has its own advantages and appears in different situations.

Consider the Hermitian model with a quadratic potential,

$$\mathcal{L} = \mathrm{tr}\left(\tfrac{1}{2}\dot{M}^2 - \omega^2 M^2\right), \tag{64}$$

with ω a scalar frequency. The above can also be written in terms of matrix elements

$$\mathcal{L} = \sum_{jk} \frac{1}{2}|\dot{M}_{jk}|^2 - \omega^2 |M_{jk}|^2. \tag{65}$$

This is nothing but N^2 harmonic oscillators. So the system is in principle trivial and solvable. All its nontrivial features emerge from the reduction to a subspace corresponding to a fixed value for the 'angular momentum' J. Reducing M to its matrix elements is not beneficial for this purpose; we need to treat it as a matrix.

We begin by defining a canonical momentum matrix conjugate to the 'coordinate' M

$$P = \frac{\partial \mathcal{L}}{\partial \dot{M}} = \dot{M}. \tag{66}$$

In terms of M and P, the Hamiltonian of the model becomes

$$H = \mathrm{tr}\left(\tfrac{1}{2}P^2 + \tfrac{1}{2}\omega^2 M^2\right). \tag{67}$$

The Poisson brackets are

$$\{M_{jk}, P_{lm}\} = \delta_{jm}\delta_{lk}. \tag{68}$$

Upon quantization, the matrix elements of M and P become operators and the above Poisson brackets become quantum mechanical commutators (not to be confused with matrix commutators). The commutator or M_{jk} and P_{lm} can be written conveniently by thinking of indices j, k as acting on a linear space 1 and indices l, m as acting on linear space 2. The 4-index symbol

$$\delta_{jm}\delta_{lk} \equiv (T_{12})_{jk;lm} \tag{69}$$

acts on both spaces and is, in fact, the operator exchanging the two spaces. Denoting with X_1 and X_2 any matrix X acting on space 1 or space 2, respectively, T_{12} satisfies

$$X_1 T_{12} = T_{12} X_2, \qquad X_2 T_{12} = T_{12} X_1. \tag{70}$$

We also note the partial trace and unimodularity properties

$$\mathrm{tr}_1\, T_{12} = I_2, \qquad \mathrm{tr}_2\, T_{12} = I_1, \qquad T_{12}^2 = I_{12}, \tag{71}$$

with I_1, I_2 and I_{12} denoting the identity matrix in space 1, space 2 and space 1×2, respectively. With the above notation, the commutator of M and P becomes

$$[M_1, P_2] = \mathrm{i} T_{12}. \tag{72}$$

The complex matrix $Q = \dot{M} + \mathrm{i}\omega M$ introduced before and its conjugate $Q^\dagger$ are the matrix analogues of creation and annihilation operators. Defining matrix operators with the standard quantum normalization

$$A^\dagger = \frac{1}{\sqrt{2\omega}}(P + \mathrm{i}\omega M), \qquad A = \frac{1}{\sqrt{2\omega}}(P - \mathrm{i}\omega M), \tag{73}$$

they satisfy

$$\left[A_1, A_2^\dagger\right] = T_{12}, \qquad [A_1, A_2] = \left[A_1^\dagger, A_2^\dagger\right] = 0. \tag{74}$$

By analogy with the standard harmonic oscillator, let us define the matrices

$$L = A^\dagger A, \qquad R' = -AA^\dagger. \tag{75}$$

The matrix operator $L = A^\dagger A$ is normal ordered, since all (quantum) creation operators are to the left of annihilation operators. R', on the other hand, is not. We can define a new (quantum) normal ordered operator R, in which the matrix multiplication is first performed in the order $AA^\dagger$ and *then* the creation matrix elements are moved to the left. Specifically,

$$R_{jk} = -:A_{js}(A^\dagger)_{sk} := (A^\dagger)_{sk}A_{js}. \tag{76}$$

Using the commutation relations of A and $A^\dagger$, we see that R and R' are trivially related:

$$R = R' + N. \tag{77}$$

In terms of L and R, the Hamiltonian can be written as

$$H = \frac{1}{2}\omega\,\mathrm{tr}(AA^\dagger + A^\dagger A) = \omega\,\mathrm{tr}\,L + \frac{N^2}{2}\omega = -\omega\,\mathrm{tr}\,R + \frac{N^2}{2}\omega. \tag{78}$$

From the basic commutators (74), we can infer the commutation relations

$$\left[L_1, A_2^\dagger\right] = T_{12}A_2^\dagger, \qquad \left[R_1, A_2^\dagger\right] = A_2^\dagger T_{12} \tag{79}$$

as well as their Hermitian conjugates. Tracing the first equation above with respect to space 1, we obtain

$$[H, A^\dagger] = \omega A^\dagger. \tag{80}$$

This means that any matrix element of A is a creation operator, creating one quantum of energy ω. This is hardly surprising, since the matrix model is, indeed, N^2 harmonic oscillators with identical frequency ω.

The commutators of the matrix elements of L and R are calculated to be

$$\begin{aligned} &[L_1, L_2] = (L_1 - L_2)T_{12}, \qquad [R_1, R_2] = (R_1 - R_2)T_{12}, \\ &[L_1, R_2] = 0. \end{aligned} \tag{81}$$

The above is nothing but two commuting copies of the $U(N)$ algebra in disguise. To see this, define the fundamental $U(N)$ generators T^a, $a = 0, 1, \ldots, N^2 - 1$, with $T^0 = I/\sqrt{N}$ satisfying the normalization condition

$$\mathrm{tr}(T^a T^b) = \delta^{ab} \tag{82}$$

and the $U(N)$ commutation relations

$$[T^a, T^b]_{\mathrm{matrix}} = \mathrm{i}f^{abc}T^c, \tag{83}$$

where $[\cdot,\cdot]_{\mathrm{matrix}}$ is a matrix commutator. Then expand L and R in the complete basis T^A of Hermitian $N \times N$ matrices:

$$L^a = \mathrm{tr}(T^a L), \qquad L = \sum_a T^a L^a \tag{84}$$

and similarly for R^a. The (scalar) expansion coefficients L^a, R^a, upon use of (81), satisfy

$$\begin{aligned} &[L^a, L^b] = \mathrm{i} f^{abc} L^c, \qquad [R^a, R^b] = \mathrm{i} f^{abc} R^c, \\ &[L^a, R^b] = 0. \end{aligned} \tag{85}$$

Since L and R mutually commute, their powers and traces commute as well, so

$$[\operatorname{tr} L^n, \operatorname{tr} R^m] = 0. \tag{86}$$

On the other hand, using the fundamental commutation relation for A and $A^\dagger$ (74) and the properties of T_{12} ((70), (71)), we can relate the traces of L and R as

$$\begin{aligned} \operatorname{tr}_1 L_1^n &= \operatorname{tr}_1 \left(A_1^\dagger A_1\right)^n = \operatorname{tr}_1 \left[A_1^\dagger \operatorname{tr}_2 T_{12} A_1 \left(A_1^\dagger A_1\right)^{n-1}\right] \\ &= \operatorname{tr}_1 \operatorname{tr}_2 \left[A_1^\dagger T_{12} A_1 \left(A_1^\dagger A_1\right)^{n-1}\right] = \operatorname{tr}_{12} \left[T_{12} A_2^\dagger A_1 \left(A_1^\dagger A_1\right)^{n-1}\right] \\ &= -\sum_{k=0}^{n-1} \operatorname{tr}_{12} \left[T_{12} \left(A_1 A_1^\dagger\right)^k T_{12} \left(A_1^\dagger A_1\right)^{n-1-k}\right] + \operatorname{tr}_{12} \left[T_{12} \left(A_1 A_1^\dagger\right)^{n-1} A_1 A_2^\dagger\right] \\ &= -\sum_{k=0}^{n-1} \operatorname{tr}(AA^\dagger)^k \operatorname{tr}(A^\dagger A)^{n-1-k} + \operatorname{tr}(AA^\dagger)^n \\ &= \sum_{k=1}^{n} (-1)^k \operatorname{tr}(R-N)^{k-1} \operatorname{tr} L^{n-k} + (-1)^n \operatorname{tr}(R-N)^n. \end{aligned} \tag{87}$$

Working recursively with the above relation, we can express traces of L entirely in terms of traces of R and vice versa; mutual commutativity of the two sets, therefore, translates into commutativity of the elements of each set, which shows that the quantities

$$I_n = \operatorname{tr}(AA^\dagger)^n \tag{88}$$

are conserved and are in involution.

The above proof may seem a bit 'too slick'. Nevertheless, the involution of I_n should be clear from (81): I_n are nothing but the Casimirs of the $U(N)$ algebra generated by L^a. $I_1 = \operatorname{tr} L \sim H$ is simply the $U(1)$ charge, while the rest are (related to) the higher $SU(N)$ Casimirs. Since $SU(N)$ is of rank $N-1$ and has $N-1$ independent Casimirs, we recover N commuting charges altogether. The higher charges I_n, $n > N$, are related to the fundamental charges $I_1, \ldots, I_N$ via nontrivial, nonlinear, N-dependent relations.

4.3. The Hermitian matrix model: reduction and spectrum

It remains to do the reduction to the 'gauge' sectors, that is, fix the charge $J = \mathrm{i}[M, \dot{M}]_{\text{matrix}}$ and work out its implications for the quantum states of the system.

The 'angular momentum' J classically generates unitary conjugation of the matrix M; this implies that quantum mechanically it will become an $SU(N)$ algebra. This can be seen explicitly: from its definition J can be expressed in terms of the operators defined in the previous section:

$$J = i : [M, P]_{\text{matrix}} :=: [A^\dagger, A]_{\text{matrix}} := A^\dagger A - : AA^\dagger := L + R. \tag{89}$$

L and R are two commuting $U(N)$ algebras, therefore their sum is another $U(N)$ algebra.

Classically, J is traceless and therefore its $U(1)$ part vanishes, making it an $SU(N)$ matrix. Quantum mechanically, this should still be true, since it generates the transformation

$$M \to UMU^{-1}, \tag{90}$$

which has trivial $U(1)$ part. This is ensured by the normal ordering of the above expression for J. L and R satisfy $\operatorname{tr} L = -\operatorname{tr} R$ and so $\operatorname{tr} J = 0$.

Reduction of the system to particular values of J corresponds, quantum mechanically, to fixing the representation of the $SU(N)$ algebra J. So the system decomposes into sectors labelled by the allowed irreducible representations (irreps) of J. Further, states within each sector related via the action of J are identified as a unique physical state, since J is a 'gauge' symmetry.

To identify these sectors, we first need to identify the possible irreps for L and R. This can be done by examining their form. Take L, at first: it is nothing but N copies of the Jordan–Wigner bosonic oscillator construction of the $U(N)$ algebra over the fundamental representation. This may require some explanation.

Let $R^a_{\alpha\beta}$ be any d-dimensional representation of a Lie group, where $\alpha, \beta = 1, \ldots, d$ label its matrix elements and a labels its generators. Define a set of d commuting creation and annihilation operators $a_\alpha, a^\dagger_\alpha$:

$$\left[a_\alpha, a^\dagger_\beta\right] = \delta_{\alpha\beta}. \tag{91}$$

The operator obtained by 'sandwiching' the matrix R^A between the vectors $a^\dagger$ and a

$$J^a = a^\dagger_\alpha R^a_{\alpha\beta} a_\beta \tag{92}$$

satisfies the commutation relations of the Lie algebra. Therefore, it provides representations of the algebra, embedded in the Fock space of the oscillators. Specifically, it provides all representations generated by the *fully symmetrized* direct product of any number of representations R. This includes the singlet (the Fock ground state), R itself (the set of d states with excitation number 1), etc.

For the specific case of the $U(N)$ Lie algebra with R the fundamental representation F, we need N oscillators and we get

$$J^a = a^\dagger_j T^a_{jk} a_k. \tag{93}$$

Transforming J from the generator basis J^a to the matrix basis $J = \sum_a J^a T^a$, J simply becomes

$$J_{jk} = a^\dagger_j a_k. \tag{94}$$

In view of the above, the matrix elements of L can be written as

$$L_{jk} = (A^\dagger)_{js} A_{sk} = A^\dagger_{sj} A_{sk}. \tag{95}$$

It is clear that j and k play the role of the fundamental indices of $U(N)$, while s is a summation index that runs over N values; for each fixed s, $A^\dagger_{sj}$ and A_{sk} play the role of $a^\dagger_j$ and a_k, respectively, while for different s all operators commute. So the above L is the direct sum of N independent (commuting) Jordan–Wigner realizations of $U(N)$ over the fundamental.

For each fixed s this realization includes, as explained earlier, all the totally symmetric tensor products of the fundamental representation of $U(N)$; that is, all irreps with a single row in their Young tableau. The direct sum of N such representations, however, includes irreps with up to N rows, which is the general case. We conclude that the spectrum of L spans the *full set* of irreps of $U(N)$. The $U(1)$ charge ($\operatorname{tr} L$) is simply the total excitation number and is given by the number of boxes of the irrep.

A Jordan–Wigner construction based on the antifundamental representation $-(T^a)^* = -(T^a)^t$ would lead to an expression in terms of N oscillators

$$J = -a^\dagger_j a_i. \tag{96}$$

Comparing with the expression for R

$$R = -A^\dagger_{ks} A_{js}, \tag{97}$$

we conclude that R is the direct sum of N independent Jordan–Wigner realizations of $U(N)$ over the antifundamental, which again spans the full set of irreps of $SU(N)$. The $U(1)$ charge, now, is the negative of the total excitation number.

Representations of L and R are, in fact, constrained to be conjugate to each other. This arises due to their construction in terms of the same bosonic creation and annihilation operators, and is also manifest by the relation between their traces, which makes all their even (odd) Casimirs equal to (minus) each other. So the representations that J can carry are of the form $r \times \bar{r}$. Such representations include the singlet, the adjoint, etc and they have as common property that their Z_N charge vanishes. We conclude that the irreps of J must have a number of boxes in their Young tableau that is *an integer multiple of N*.

The quantum states of the model can be constructed by starting with the ground state $|0\rangle$, annihilated by all operators A_{jk}

$$A_{jk}|0\rangle = 0 \tag{98}$$

and acting with any number of creation operators:

$$|j_1, k_1; j_2, k_2; \ldots; j_n, k_n\rangle = A^\dagger_{j_1 k_1} A^\dagger_{j_2 k_2} \cdots A^\dagger_{j_n k_n}|0\rangle. \tag{99}$$

For any c-number (classical) matrix Φ, the commutation relations

$$\begin{aligned} &[\mathrm{tr}(\Phi L), A^\dagger] = \Phi A^\dagger, \qquad [\mathrm{tr}(\Phi R), A^\dagger] = -A^\dagger \Phi, \\ &[\mathrm{tr}(\Phi J), A^\dagger] = \Phi A^\dagger - A^\dagger \Phi \end{aligned} \tag{100}$$

imply that the first (left) index of the matrix $(A^\dagger)_{jk}$ transforms in the fundamental (F) under L, the second (right) index transforms in the antifundamental ($\bar{F}$) under R and that $A^\dagger$ altogether transforms in the adjoin under J. The general state transforms in a representation of the form $F \times F \times \cdots \times \bar{F} \times \bar{F} \times \cdots$ with n Fs and n $\bar{F}$s appearing, corresponding the number of free left and right indices in the state.

Physical states are chosen by imposing the constraint that J is in a fixed representation of $SU(N)$, say r. To do this, we start from a generic state $|j_1, k_1; j_2, k_2; \ldots; j_n, k_n\rangle$ and contract the free indices with Clebsch–Gordan coefficients that project it to this representation. All states within the same representation r are gauge equivalent and therefore represent a unique physical state. States corresponding to inequivalent copies of r, however, contained in the representation of the initial state correspond to distinct physical states.

The above analysis can be done explicitly in the case of the representation r corresponding to the standard (spinless) Calogero model. Classically, the matrix commutator $\mathrm{i}[M, \dot{M}] = J$ is of the form $\ell(vv^\dagger - 1)$, where v is a vector of length squared equal to N. The representation r_ℓ corresponding to this is the fully symmetric one with a number of Young tableau boxes equal to ℓN.

The easiest way to realize this is by quantizing the classical expression $J = \ell vv^\dagger - \ell = \psi\psi^\dagger - \ell$ promoting the vector components $\psi_j \ell^{\frac{1}{2}} v_j$ to yet another set of harmonic oscillator creation operators

$$\left[\psi_j, \psi_k^\dagger\right] = \delta_{jk}. \tag{101}$$

In this way, the operator

$$(J_\psi)_{jk} = -\psi_k^\dagger \psi_j \tag{102}$$

realizes the $U(N)$ algebra, again in a Jordan–Wigner construction over the antifundamental. Its representation content includes all fully symmetric products of the antifundamental, each of them represented in the subspace of fixed excitation number $n = \psi_j^\dagger \psi_j$. The restriction of J to the symmetric representation r_ℓ can be expressed by the condition

$$J + J_\psi + \ell = 0. \tag{103}$$

This means that the representations carried by J and J_ψ must be conjugate to each other so that their sum (composition) contain the identity, which fixes the representation of J to be a totally symmetric one. The c-number term ℓ serves the purpose of subtracting the trace of J_ψ, which is required by the tracelessness of J. It also fixes the $U(1)$ part of J_ψ, since tracing the above relation gives

$$\operatorname{tr} J_\psi + \ell N = -\sum_j \psi_j^\dagger \psi_j + \ell N = 0. \tag{104}$$

We recover the condition that $n = \ell N$ is the total number of boxes in the Young tableau of J_ψ and thus also of J, fully fixing the desired representation.

As we stated earlier, representations of J must have a number of Young tableau boxes that is an integer multiple of N. An important corollary of the above analysis, therefore, is that ℓ must be quantized to an integer. This is a new feature of the Calogero system as deriving from the matrix model. It is *not* a general requirement for the quantum Calogero model, which is perfectly well defined for fractional values of ℓ. The source of this quantization is the enlargement of the symmetry group of the system from S_N to $SU(N)$ for the matrix model: the enlarged symmetry has a global 'anomaly' which requires the quantization of the coupling constant ℓ, in analogy with similar effects in gauge theory.

Finally, states of the theory can now be constructed in terms of the vacuum state annihilated by A and ψ upon the action of $\psi^\dagger$ and $A^\dagger$. The full state must be a singlet under $J + J_\psi$, which means that all indices, *including those of* $\psi^\dagger$, must be contracted. Gauge-invariant contraction of indices can be done by matrix multiplication, but tracing or by multiplication of the vector $\psi^\dagger$ and the matrix $A^\dagger$.

Each $\psi^\dagger$ in the state, therefore, will leave one index hanging, since there is no way to contract it with another $\psi^\dagger$ and multiplication with $A^\dagger$ still leaves one uncontracted index. The only way to contract these indices is using the only invariant tensor of $SU(N)$ that includes all fundamental indices, namely the N-fold antisymmetric tensor $\epsilon_{j_1 \dots j_N}$. This means that $\psi^\dagger$s must come in multiples of N, recovering once more the condition that the total excitation number $n = \ell N$ must be a multiple of N. Further, operators contracted with ϵ must all be distinct, otherwise the product would vanish due to antisymmetry.

The generic form of the physical states is [24]

$$[\operatorname{tr} A^\dagger]^{m_1} [\operatorname{tr}(A^\dagger)^2]^{m_2} \cdots [\operatorname{tr}(A^\dagger)^N]^{m_N} \left[\epsilon_{j_1 \dots j_N} \psi^\dagger_{j_1} (\psi^\dagger A^\dagger)_{j_2} \cdots (\psi^\dagger (A^\dagger)^{N-1})_{j_N}\right]^\ell |0\rangle. \tag{105}$$

Higher traces of $A^\dagger$ can be related to the first N ones, and other, more general ways of contracting operators with ϵ tensors can be reduced to linear combinations of the above states.

This is an eigenstate of the Hamiltonian $H = \omega \operatorname{tr}(A^\dagger A) + \omega N^2/2$ with its energy given by the total number of $A^\dagger$ oscillators appearing in the state plus a constant. The ψ-dependent prefactor contributes an energy equal to $\omega \ell N(N-1)/2$. We conclude that the energy spectrum is

$$E = \omega \left(\sum_{k=1}^{N} m_k + \ell \frac{N(N-1)}{2} + \frac{N^2}{2} \right). \tag{106}$$

We recognize the excitation energies as those of N non-interaction bosons in a harmonic oscillator potential, expressed in terms of their collective excitations (m_k represents the energy gap between the top k bosons and the next lower one on the single-particle oscillator spectrum). By standard bosonization arguments, the same excitation spectrum applies to a set of N non-interacting fermions in a harmonic oscillator ($m_k + 1$ represents the energy gap between the

top k fermions and the next lower one on the single-particle oscillator spectrum). So we may rewrite the above spectrum as

$$E = \omega\left(\sum_{i=1}^{N} n_i + \ell\frac{N(N-1)}{2} + \frac{N^2}{2}\right) = \omega\left(\sum_{i_1}^{N} \bar{n}_i + \frac{N}{2}\right), \tag{107}$$

where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_N$ are single-particle bosonic excitation numbers while the 'pseudoexcitation' numbers $\bar{n}_i$ have been defined as

$$\bar{n}_i = n_i + (\ell+1)(i-1). \tag{108}$$

We recover the spectrum of the Calogero model as exposed in a previous section, with the extra shift $\ell \to \ell + 1$. This is a quantum shift of the 'bare' parameter ℓ appearing in the classical matrix model to the renormalized value $\ell + 1$ entering the Calogero model. In particular, the single sector $\ell = 0$ of the matrix model corresponds to $\ell = 1$ in the Calogero model, that is, fermions. The 'fermionization' of the eigenvalues of the matrix model due to the quantum mechanical measure arising out of integrating out the angular variables of the matrix is a well-known effect.

Other, more general representations of J can be dealt with in similar ways. As shall be explained later, they correspond to Calogero particles with internal degrees of freedom ('spin'). We shall examine these cases in the context of the unitary matrix model. We have analysed quite enough already the Hermitian matrix model and we should leave something interesting for the unitary case!

5. The unitary matrix model

5.1. Classical analysis

The Hermitian matrix model works well for particles on the line but has trouble representing particles on periodic spaces. The most natural candidate for such models would be a unitary $N \times N$ matrix U. Its eigenvalues are phases and naturally live on the circle. We start, therefore, with a Lagrangian that represents the invariant kinetic energy on the space of such matrices:

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{tr}(U^{-1}\dot{U})^2. \tag{109}$$

A potential could in principle be included but we are interested in the translationally invariant case and will omit it. The treatment is similar as before, and we just summarize the relevant facts.

The Lagrangian is, in fact, invariant under separate left and right multiplications of U by time-independent unitary matrices and there are two corresponding conserved matrix angular momenta L and R:

$$U \to VU: \qquad L = \mathrm{i}\dot{U}U^{-1}, \tag{110}$$

$$U \to UW^{-1}: \qquad R = -\mathrm{i}U^{-1}\dot{U}. \tag{111}$$

The unitary conjugation that preserves the eigenvalues corresponds to $W = V$ and its generator is

$$J = L + R = \mathrm{i}[\dot{U}, U^{-1}]. \tag{112}$$

The rest of the discussion is as previously. Parametrizing

$$U = V\Lambda V^{-1} \qquad \text{with} \qquad \Lambda = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}x_i}, \ldots, \mathrm{e}^{\mathrm{i}x_N}\}, \tag{113}$$

the Hamiltonian becomes, after a few steps,

$$H = \sum_i \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{K_{ij} K_{ji}}{4 \sin^2 \frac{x_i - x_j}{2}}, \tag{114}$$

where, as before,

$$K = V^{-1} J V. \tag{115}$$

Choosing $J = K = 0$ reproduces free particles on the circle, while choosing $J = \ell(uu^\dagger - 1)$ we obtain $K_{ij} K_{ji} = \ell^2$ and we recover the Sutherland inverse-sine-square model

$$H = \sum_i \frac{1}{2} \dot{x}_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{\ell^2}{4 \sin^2 \frac{x_i - x_j}{2}}. \tag{116}$$

This model is integrable and solvable by the same techniques as the Hermitian one. The conserved invariant quantities are

$$I_n = \operatorname{tr} L^n = \operatorname{tr}(-R)^n = \operatorname{tr}(\mathrm{i} U^{-1} \dot{U})^n \tag{117}$$

and the solution is

$$U = B \, \mathrm{e}^{\mathrm{i} C t} \tag{118}$$

with B a unitary and C a Hermitian matrix satisfying

$$BCB^{-1} - C = J. \tag{119}$$

For the Sutherland case with $J = \ell(uu^\dagger - 1)$, $u_i = 1$, B, C become

$$B_{jk} = \delta_{jk} \, \mathrm{e}^{\mathrm{i} q_j}, \qquad C_{jk} = \delta_{jk} p_j + (1 - \delta_{jk}) \frac{\mathrm{i}\ell}{\mathrm{e}^{\mathrm{i}(q_j - q_k)} - 1}, \tag{120}$$

where, clearly, q_i and p_i are initial positions and momenta.

We conclude by mentioning that, upon scaling $x \to \alpha x$, $t \to \alpha^2 t$, the Sutherland model goes over to the free Calogero model. This is the 'infinite volume' limit.

5.2. Unitary matrix model: quantization

The quantization of the unitary matrix model can be performed in a way practically identical to the Hermitian model. Indeed, it should be obvious that the matrices L and R defined in the two models have identical properties and the whole analysis can be done in a parallel way. The new element of the unitary model is that the Hamiltonian is proportional to the quadratic Casimir $I_2 = \operatorname{tr} L^2$, rather than the linear one, and appropriate bases of states have to be found in the degenerate spaces of $I_1 = \operatorname{tr} L$ that diagonalize also I_2.

We shall, however, give an independent treatment of the unitary model. The reasons are primarily pedagogical: we shall use an explicit index notation, rather than the neat T_{12} calculus of the previous sections, just to familiarize ourselves with the alternative. Further, we shall make specific use of the connection of the $N \times N$ unitary model with the group $U(N)$, taking advantage of (and making contact with) known representation facts for this group.

We begin, again, by defining a canonical momentum matrix conjugate to the 'coordinate' U:

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{U}} = -U^{-1} \dot{U} U^{-1}. \tag{121}$$

The Poisson brackets are

$$\{U_{jk}, \Pi_{lm}\} = \delta_{jm} \delta_{lk}. \tag{122}$$

Π is somewhat unpleasant, being neither unitary nor Hermitian. We prefer to work in terms of the Hermitian matrices L and R defined previously

$$L = \mathrm{i}\dot{U}U^{-1} = -\mathrm{i}U\Pi, \qquad R = -\mathrm{i}U^{-1}\dot{U} = \mathrm{i}\Pi U. \tag{123}$$

Using (122) we derive the following Poisson brackets:

$$\{L_{jk}, L_{lm}\} = \mathrm{i}(L_{jm}\delta_{lk} - \delta_{jm}L_{lk}), \tag{124}$$

$$\{L_{jk}, R_{lm}\} = 0, \tag{125}$$

$$\{R_{jk}, R_{lm}\} = \mathrm{i}(R_{jm}\delta_{lk} - \delta_{jm}R_{lk}). \tag{126}$$

The above is nothing but two copies of the $U(N)$ algebra in disguise. To see this, expand the matrices L and R in the basis of the fundamental generators of $SU(N)T^a$ plus the unit matrix:

$$L = L^o + 2\sum_{a=1}^{N^2-1} L^a T^a, \tag{127}$$

$$R = R^o + 2\sum_{a=1}^{N^2-1} R^a T^a, \tag{128}$$

with L^o, L^a, R^o, R^a numbers. Then use the $SU(N)$ commutation relations

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c \tag{129}$$

as well as the normalization

$$\mathrm{tr}(T^a T^b) = \tfrac{1}{2}\delta_{ab} \tag{130}$$

to show that the expansion coefficients satisfy the Poisson algebra

$$\{L^a, L^b\} = f^{abc}L^c, \tag{131}$$

$$\{L^a, R^b\} = 0, \tag{132}$$

$$\{R^a, R^b\} = f^{abc}R^c, \tag{133}$$

while L^o, R^o are central. Note that the $U(1)$ charges

$$L^o = -R^o = \frac{1}{N}\,\mathrm{tr}(-\mathrm{i}U^{-1}\dot{U}) = \frac{1}{N}\sum_i \dot{x}_i \tag{134}$$

are essentially the total momentum of the system.

We are now ready to perform quantization. In the U representation, where states are functions of U, Π becomes the matrix derivative $\Pi_{jk} = -\mathrm{i}\delta_U$, acting as

$$\delta_U\,\mathrm{tr}(UB) = B, \qquad \delta_U\,\mathrm{tr}(U^{-1}B) = -U^{-1}BU^{-1}, \tag{135}$$

where B is a constant matrix, and similarly for expressions containing more Us. L and R, upon proper ordering, are represented as

$$L = -U\delta_U, \qquad R = \delta_U \cdot U, \tag{136}$$

where in R it is understood that we *first* act with the derivative and *then* right multiply the result by U. With this ordering, L and R become the proper $U(N)$ operators acting as

$$L\,\mathrm{tr}(UB) = -UB, \qquad L\,\mathrm{tr}(U^{-1}B) = BU^{-1}, \tag{137}$$

$$R\,\mathrm{tr}(U\,B) = BU, \qquad R\,\mathrm{tr}(U^{-1}B) = -U^{-1}B. \tag{138}$$

It is also useful to express their action on arbitrary functions of U as

$$\mathrm{tr}(\mathrm{i}\epsilon L)\,f(U) = f((1-\mathrm{i}\epsilon)U) - f(U), \tag{139}$$

$$\mathrm{tr}(\mathrm{i}\epsilon R)\,f(U) = f(U(1+\mathrm{i}\epsilon)) - f(U), \tag{140}$$

where ϵ is an arbitrary infinitesimal Hermitian matrix, emphasizing their role as generators of left and right multiplication on U. Correspondingly, the operators L^a and R^a satisfy the $SU(N)$ algebra. Their action can be obtained by taking $\epsilon = \varepsilon T^a$ with ε an infinitesimal scalar parameter, that is,

$$\mathrm{i}\varepsilon L^a f(U) = f((1-\mathrm{i}\varepsilon T^a)U) - f(U), \tag{141}$$

$$\mathrm{i}\varepsilon R^a f(U) = f(U(1+\mathrm{i}\varepsilon T^a)) - f(U). \tag{142}$$

The Hamiltonian, being classically the kinetic term on the manifold of unitary matrices $U(N)$, quantum mechanically becomes the Laplacian operator on the manifold [30]. Using (127), (128) it is expressed as

$$H = \frac{1}{2}\,\mathrm{tr}\,L^2 = \sum_a (L^a)^2 + \frac{1}{2}N(L^o)^2 = \sum_a (R^a)^2 + \frac{1}{2}(R^o)^2 = \frac{1}{2}\,\mathrm{tr}\,R^2. \tag{143}$$

It is, therefore, the common quadratic Casimir of the left and right $SU(N)$ algebras plus the square of the $U(1)$ charge, the two parts identifiable as the relative and centre-of-mass energy, respectively.

Quantum mechanical states grouping into irreducible representations of the L and R $SU(N)$ algebras will, thus, be degenerate multiplets of the Hamiltonian. The $U(1)$ (centre-of-mass) part trivially separates: we can boost any state by any desired total momentum NP by multiplying the wavefunction by $(\det U)^P$. We will examine only the $SU(N)$ part from now on.

A natural basis of states for the Hilbert space is the matrix elements of the unitary irreducible representations (irreps) of $SU(N)$. Let R denote such an irrep, $R(U)$ the matrix that represents U in this irrep and $R_{\alpha\beta}(U)$ the $\alpha\beta$ matrix element of this matrix. Clearly, α and β range from 1 to the dimensionality of R, d_R. $R_{\alpha\beta}(U)$ are a complete orthonormal basis of wavefunctions for U, that is

$$\int [\mathrm{d}U] R_{\alpha\beta}(U) R'_{\gamma\delta}(U)^* = \delta_{RR'}\delta_{\alpha\gamma}\delta_{\beta\delta} \tag{144}$$

with $[\mathrm{d}U]$ the volume element on the space of $SU(N)$ matrices as implied by the metric $\mathrm{d}s^2 = -\,\mathrm{tr}(U^{-1}\,\mathrm{d}U)^2$, also called the Haar measure.

We will, now, show that each $R_{\alpha\beta}(U)$ is an eigenstate of the Hamiltonian with eigenvalue equal to the quadratic Casimir of R, C_R. Qualitatively, after the discussion of the last paragraphs, this should be obvious: L and R generate the transformations $U \to V^{-1}U$ and $U \to UW$. $R(U)$ transforms in the conjugate irrep $\bar{R}$ under L and in the irrep R under R. Since H is the common quadratic Casimir of L and R, we conclude that all d_R^2 states $R_{\alpha\beta}(U)$ are energy eigenstates with eigenvalue $C_R = C_{\bar{R}}$.

If you are confused about L generating $U \to V^{-1}U$ rather than $U \to VU$, think of the difference between active and passive transformations, which is relevant when shifting from classical to quantum: $\psi(x-a)$ shifts the wavefunction by $+a$. Also, although classical transformations on U compose properly,

$$V_1(V_2U) = (V_1V_2)U, \tag{145}$$

quantum mechanically the operators $\hat{V}$ that perform the shift $U \to VU$ on the argument of the wavefunction would compose

$$\hat{V}_1(\hat{V}_2 f(U)) = \hat{V}_1 f(V_2 U) = f(V_2 V_1 U) = (\hat{V}_2 \hat{V}_1) f(U). \tag{146}$$

Therefore, we need to invert the action of $\hat{V}$ to get the right composition law.

Let us prove the fact $H R_{\alpha\beta}(U) = C_R R_{\alpha\beta}(U)$ more analytically. Since $R(U)$ is a representation, it obeys the group property

$$R_{\alpha\beta}(UV) = \sum_\gamma R_{\alpha\gamma} R_{\gamma\beta}(V). \tag{147}$$

From (141) we have

$$(1 + \mathrm{i}\varepsilon) L^a R_{\alpha\beta}(U) = R_{\alpha\beta}((1 - \mathrm{i}\varepsilon T^a)U) = R_{\alpha\gamma}(1 - \mathrm{i}\varepsilon T^a) R_{\gamma\beta}(U) \tag{148}$$

$$= R_{\alpha\beta}(U) - \mathrm{i}\varepsilon R^a_{\alpha\gamma} R_{\gamma\beta}(U), \tag{149}$$

where $R^a = R(T^a)$ is the ath generator of $SU(N)$ in the R representation. So

$$L^a R_{\alpha\beta}(U) = -R^a_{\alpha\gamma} R_{\gamma\beta}(U) \tag{150}$$

and

$$\sum_a (L^a)^2 R_{\alpha\beta}(U) = \sum_a R^a_{\alpha\gamma} R^a_{\gamma\delta} R_{\delta\beta}(U) = \sum_a (R^a)^2_{\alpha\delta} R_{\delta\beta}(U). \tag{151}$$

The sum $\sum_a (R^a)^2$ appearing above is the quadratic Casimir in the irrep R and is proportional to the identity matrix $\delta_{\alpha\delta}$. So, finally,

$$H R_{\alpha\beta}(U) = C_R R_{\alpha\beta}(U). \tag{152}$$

Incidentally, the spectrum spanned by C_R for all R is nothing but the spectrum of N free fermions on the circle with the ground-state energy and the centre-of-mass energy subtracted, where the lengths R_i of the rows of the Young tableau of R correspond to the 'bosonized' fermion momenta

$$p_i = R_i - i + 1 \tag{153}$$

and where the centre-of-mass energy has been subtracted. The condition $R_i \geqslant R_{i+1}$ for the rows amounts to the fermionic condition $p_i > p_{i+1}$. The spectrum of the full matrix model, then, is identical to the free fermion one but with *different degeneracies*.

We have, therefore, identified all energy eigenstates of the matrix model. It remains to implement the quantum analogue of the choice of angular momentum J, identify the corresponding reduced quantum model and pick the subspace of states of the full model that belongs to the reduced model.

J obeys itself the $SU(N)$ algebra (it is traceless, no $U(1)$ charge). As in the Hermitian model, a choice of value for J amounts to a choice of irrep r for this algebra. States within the same irrep are related by unitary transformations of U and give the same dynamics; they correspond to a unique physical state. Since $J = L + R$, we see that states transforming under (L, R) in the $(\bar{R}, R)$ irreps will transform in $\bar{R} \times R$ under J. So, only irreps r that are contained in the direct product of two mutually conjugate irreps can be obtained for J. This amounts to irreps r with a number of boxes in their Young tableau that is an integer multiple of N, just as in the Hermitian model case. (To get a feeling of this, consider the case $N = 2$. Then J is an orbital-like realization of the angular momentum through derivatives of U and clearly cannot admit spinor representations.)

We must, therefore, project the d_R^2 states in $R_{\alpha\beta}(U)$ to the subspace of states transforming as r under $L+R$. Call $G(\bar{R},\alpha;\,R,\,\beta|r,\,\gamma)$ the Clebsch–Gordan coefficient that projects these states to the γ state of r. Then the relevant states for this model become

$$\Psi_R(U)=\sum_{\alpha,\beta}R_{\alpha\beta}(U)G(\bar{R},\alpha;\,R,\,\beta|r,\,\gamma). \tag{154}$$

The index γ labelling the states within r, as we argued before, counts the d_r gauge copies and does not imply a true degeneracy of states. The degeneracy of the states produced by each R is, then, given by the number of times the irrep r is contained in the direct product $\bar{R}\times R$ or, equivalently, the number of times R is contained in $R\times r$. Calling this integer $D(R,r;\,R)$, we obtain for the spectrum and degeneracies:

$$E_R=C_R,\qquad D_R=D(R,r;\,R). \tag{155}$$

In particular, if $D_R=0$ the corresponding energy level is absent from the spectrum.

Concluding, we mention that an approach which also reproduces the spectrum and states of the Sutherland model is two-dimensional Yang–Mills theory on the circle [31, 32]. This approach is essentially equivalent to the matrix model above and we will not be concerned with it.

5.3. Reduction to spin–particle systems

So we have derived the spectrum, degeneracy and wavefunctions of the matrix model restricted to the sector $J=r$. Classically, these restrictions represented free particles ($J=0$), Sutherland particles ($J=\ell(vv^\dagger-1)$) or something more general. What are the corresponding quantum systems?

To find these, let us reproduce here the expression of the reduced Hamiltonian in one of these sectors:

$$H=\sum_i\frac{1}{2}p_i^2+\frac{1}{2}\sum_{i\neq j}\frac{K_{ij}K_{ji}}{4\sin^2\frac{x_i-x_j}{2}}-E_0. \tag{156}$$

This expression remains valid quantum mechanically upon a proper definition (ordering) of the operator K. The only residual quantum effect is a constant term E_0 that comes from the change of measure from the matrix space to the space of eigenvalues.

Let us expand a bit on this without entering too deeply into the calculations. (For details see, e.g., [33].) The Haar measure in terms of the diagonal and angular part of U has the form

$$[\mathrm{d}U]=\Delta^2[\mathrm{d}V], \tag{157}$$

where $[\mathrm{d}V]$ is the Haar measure of V and Δ is the Vandermonde determinant

$$\Delta=\prod_{i<j}2\sin\frac{x_i-x_j}{2}. \tag{158}$$

To see this, write the 'line element' $-\mathrm{tr}(U^{-1}\,\mathrm{d}U)^2$ in terms of V and x_i using (113) and obtain

$$-\mathrm{tr}(U^{-1}\,\mathrm{d}U)^2=\sum_i\mathrm{d}x_i^2-\sum_{i,j}4\sin^2\frac{x_i-x_j}{2}(V^{-1}\,\mathrm{d}V)_{ij}(V^{-1}\,\mathrm{d}V)_{ji}. \tag{159}$$

This metric is diagonal in $\mathrm{d}x_i$ and $(V^{-1}\,\mathrm{d}V)_{ij}$. The square root of the determinant of this metric, which gives the measure (volume element) on the space, is clearly Δ^2 times the part coming from V which is the standard Haar measure for V. (We get *two* powers of $4\sin^2\frac{x_i-x_j}{2}$ in the determinant, one from the real and other from the imaginary part of $(V^{-1}\,\mathrm{d}V)_{ij}$, so the square root of the determinant has one power of Δ^2.)

To bring the kinetic x_i-part into a 'flat' form (plain second derivatives in x_i), we must multiply the wavefunction with the square root of the relevant measure (compare with the change from Cartesian to spherical coordinates in central potential problems). The net result is that the wavefunction Ψ in terms of x_i and V is the original wavefunction $\psi(U)$ of the matrix model times the Vandermonde determinant. This, however, also produces an additive constant E_0 which comes from the action of the entire x_i-kinetic operator on Δ. Noting that Δ is nothing but the ground-state wavefunction of N free fermions on the circle, we see that E_0 is the relevant fermionic ground-state energy

$$E_0 = \frac{N(N^2-1)}{24}. \tag{160}$$

This is the famous 'fermionization' of the eigenvalues produced by the matrix model measure.

To determine the proper ordering for K, we examine its properties as a generator of transformations. Since $U = V\Lambda V^{-1}$, and J generates $U \to V'UV'^{-1} = (V'V)\Lambda(V'V)^{-1}$, we see that J generates left multiplications of the angular part V of U. $K = V^{-1}JV$, on the other hand, generates *right* multiplications of V, as can be seen from its form or by explicit calculation through its Poisson brackets. As a result, it also obeys the $SU(N)$ algebra. Its proper quantum definition, then, is such that it satisfies, as an operator, the $SU(N)$ algebra. It clearly commutes with the diagonal part x_i and its momentum p_i, since it has no action on it. Its dynamics are fully determined by the Hamiltonian (156) and its $SU(N)$ commutation relations.

We can, therefore, in the context of the particle model (156), forget where K came from and consider it as an independent set of dynamical $SU(N)$ operators. K, however, obeys some constraints. The first is that, as is obvious from $K = V^{-1}JV$, K carries *the same irrep r* as J. The second is subtler: a right multiplication of V with a diagonal matrix will clearly leave $U = V\Lambda V^{-1}$ invariant. Therefore, this change of V has no counterpart on the 'physical' degrees of freedom of the model and is a gauge transformation. As a result, we get the 'Gauss' law' that physical states should remain invariant under such transformations. Since K generates right multiplications of V, and K_{ii} (no sum) generates the diagonal ones, we finally obtain

$$\text{(no sum)} \quad K_{ii} = 0 \qquad \text{(on physical states)}. \tag{161}$$

(A more pedestrian but less illuminating way to see it is: $J = \mathrm{i}[U^{-1}, \dot{U}]$, being a commutator, vanishes when sandwiched between the same eigenstate of U. Since K is essentially J in the basis where U is diagonal, its diagonal elements vanish.) Note that the constraint (161) is preserved by the Hamiltonian (156).

The above fully fixes the reduced model Hilbert space as the product of the N-particle Hilbert space times the d_r-dimensional space of K, with the constraint (161) also imposed. The further casting of the model into something with a more direct physical interpretation relies upon a convenient realization of K. *Any* such realization will do: simply break the representation of $SU(N)$ that it carries into irreps r and read off the spectrum for each r from the results of the previous section.

We shall implement K in a construction *à la* Jordan–Wigner. Let $a_{mi}, a^\dagger_{mi}, m = 1, \ldots, q, i = 1, \ldots, N$, be a set of Nq independent bosonic oscillators [32]:

$$\left[a_{mi}, a^\dagger_{nj}\right] = \delta_{mn}\delta_{ij}. \tag{162}$$

Then

$$K^a = \sum_{m=1}^{q} a_{mi} T^a_{ij} a_{mj} \tag{163}$$

is a realization of the $SU(N)$ algebra. (T^a_{ij} are the matrix elements of T^a.) The corresponding matrix elements of K are

$$K_{ij} = \sum_{m=1}^{q} \left\{ a^\dagger_{mi} a_{mj} - \frac{1}{N} \left(\sum_k a^\dagger_{mk} a_{mk} \right) \delta_{ij} \right\}. \tag{164}$$

Correspondingly, the coefficient of the Sutherland potential in (156) is (for $i \neq j$)

$$K_{ij} K_{ji} = \sum_{m,n} a^\dagger_{mi} a_{ni} a^\dagger_{nj} a_{mj} + \sum_m a^\dagger_{mi} a_{mi}. \tag{165}$$

We already see that the degrees of freedom of K are redistributed into degrees of freedom for each particle in the above. Specifically, defining

$$S_{i,mn} = a^\dagger_{mi} a_{ni} - \frac{1}{q} \left(\sum_{s=1}^{q} a^\dagger_{si} a_{si} \right) \delta_{mn} \tag{166}$$

and comparing with (164), we see that S_i are N independent sets of operators each satisfying the $SU(q)$ algebra. Before expressing $K_{ij}K_{ji}$ in terms of S_i, let us see what the constraint (161) implies:

$$K_{ii} = \sum_{m=1}^{q} a^\dagger_{mi} a_{mi} - \frac{1}{N} \sum_{m,k} a^\dagger_{mk} a_{mk} = 0. \tag{167}$$

$\sum_{m,k} a^\dagger_{mk} a_{mk}$ commutes with all K_{ij} and all $S_{i,mn}$. It is, therefore, a Casimir and can be chosen as a fixed integer ℓN equal to the total number operator of the subspace of the oscillator Fock space in which the model lives. The above constraint, then, implies

$$\sum_{m=1}^{q} a^\dagger_{mi} a_{mi} = \ell. \tag{168}$$

(We see why we had to choose the total number operator to be a multiple of N: the operator in (168) above is also a number operator and can have only integer eigenvalues.) Using this in (166) we can express

$$a^\dagger_{mi} a_{ni} = S_{i,mn} + \frac{\ell}{q} \delta_{mn} \tag{169}$$

and therefore

$$K_{ij} K_{ji} = \sum_{mn} S_{i,mn} S_{j,nm} + \frac{\ell(\ell+q)}{q} = \vec{S}_i \cdot \vec{S}_j + \frac{\ell(\ell+q)}{q}, \tag{170}$$

where $\vec{S}_i \cdot \vec{S}_i = \mathrm{tr}(S_i S_j)$ is the $SU(q)$-invariant scalar product of the two $SU(q)$ 'vectors.' We finally obtain the Hamiltonian as [32]

$$H = \sum_i \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{2\vec{S}_i \cdot \vec{S}_j + \frac{\ell(\ell+q)}{q}}{4 \sin^2 \frac{x_i - x_j}{2}}. \tag{171}$$

So it is a Sutherland-like model but where the particles also carry $SU(q)$ internal degrees of freedom ('spins') and the potential contains a pairwise antiferromagnetic interaction between the spins.

It remains to specify the representation in which the $SU(q)$ spins are and find the irreps contained in this realization of K, therefore obtaining the spectrum. A realization of the form (164) for $q = 1$ in terms of bosonic oscillators contains all *totally symmetric* irreps of $S(N)$ (that is, those with a single row in their Young tableau). (164) is essentially the

direct product of q such independent realizations, so it contains all direct products of q totally symmetric irreps. This produces all irreps with up to q rows in their Young tableau, perhaps more than once each. The constraint (168), however, implies that the total number of boxes in the Young tableau of these irreps is ℓN. We recover once more the constraint that we derived before based on the origin of r as a component of $\bar{R} \times R$.

Similarly, the realization (166) of S_i contains all the totally symmetric irreps of $SU(q)$. (168) implies that the number of boxes of these irreps is equal to ℓ, so the spins S_i are each in the ℓ-fold symmetric irrep of $SU(q)$. Solving this model amounts to decomposing the tensor product of these N spins into irreducible components of $SU(q)$. Each such component corresponds to a subspace of the Hilbert space with a fixed total spin S. This same irrep, interpreted as an irrep r of $SU(N)$, will be the corresponding irrep of K, and also of J, and thus will determine the spectrum of this sector through (155).

Let us elucidate the above by reproducing the two simplest cases: free particles and (spinless) Sutherland particles, comparing with the classical treatment.

(a) Free particles correspond to $J = K = 0$. So there is no spin and no potential and we have non-interacting particles. From (155) we see that all D_R are one, and thus the spectrum is the free fermion one, as commented before. The matrix model naturally quantizes free particles as fermions.

(b) Spinless Sutherland particles correspond, classically, to $J = \ell(vv^\dagger - 1)$. So J is rank 1 (ignoring the trace). Quantum mechanically, this corresponds to the irrep r of J having only one row and therefore only one independent Casimir. Since q in the above realization corresponds to the number of rows, we must have $q = 1$. Spins, therefore, are absent. The strength of the potential becomes $\ell(\ell + 1)$, where ℓN is the number of boxes in one row of r. By standard Young tableau rules we see that the degeneracy D_R is 1 if the row lengths of R satisfy

$$R_i \geqslant R_{i+1} + \ell, \tag{172}$$

else it is zero. The spectrum of this model is, then, the same as the spectrum of free particles but with the selection rule for their momenta

$$p_i \geqslant p_{i+1} + \ell + 1. \tag{173}$$

We recover the 'minimum distance' selection rule of the Calogero model that led to the interpretation as particles with generalized statistics! Only, in this case, the statistics parameter $\ell + 1$ is a positive integer. The wavefunctions of the Sutherland model can be related to characters of the $U(N)$ group (see e.g. [34]).

We mention here that a Jordan–Wigner realization of K in terms of *fermionic* oscillators is also useful and leads to particles with spins interacting via *ferromagnetic* Sutherland-type potentials. The Hamiltonian becomes [32]

$$H = \sum_i \frac{1}{2} p_i^2 - \frac{1}{2} \sum_{i \neq j} \frac{2\vec{S}_i \cdot \vec{S}_j + \frac{\ell(\ell-q)}{q}}{4 \sin^2 \frac{x_i - x_j}{2}}, \tag{174}$$

where now the spins are in the ℓ-fold *anti*symmetric irrep of $SU(q)$. We will not elaborate further and leave the details as an exercise to the reader.

Clearly there are other, more general realizations, involving a mixture of fermionic and bosonic oscillators. Such a construction, e.g., involving a single bosonic and a single fermionic oscillator produces the so-called supersymmetric Calogero model [35]. The model of [36] also falls into this category.

We close this section by noting that the unitary matrix model can be thought of as motion on a space of constant positive curvature $U(N)/G$, where G is the symmetry group that leaves the angular momentum J (with which we reduce the system) invariant. For the

spinless Sutherland model, we have $G = U(N-1) \times U(1)$, where $U(N-1)$ rotates in the $N-1$ directions normal to the vector v and $U(1)$ is the phase arbitrariness of v. The spin systems correspond to more general choices of G. A similar construction for spaces of constant negative curvature leads to systems with a hyperbolic sine-squared interaction [37]. Such hyperbolic systems can be easily obtained by analytically continuing the variables x_i and p_i to the imaginary axis, which leaves the hamiltonian real.

5.4. Concluding remarks about the matrix model approach

In conclusion, the matrix model has provided us with the following:

(1) An augmentation of the permutation group into the $SU(N)$ group and a corresponding possibility of defining statistics through the irreps of $SU(N)$.
(2) A realization of generalized scalar statistics but with *quantized* statistics parameter $\ell + 1$ in terms of the Calogero model.
(3) A realization of generalized 'non-Abelian statistics' in terms of particles with internal degrees of freedom interacting through a generalized Calogero-type potential.
(4) A systematic way of solving the above models.

What the matrix models has *not* provided is

(1) A realization of the Calogero model for fractional values of the interaction parameter ℓ.
(2) A realization of spin-Calogero systems with the spins in arbitrary (non-symmetric) representations.
(3) A control of the coupling strength of the potential for the spin-Calogero models. (Note that the coefficient of $\vec{S}_i \cdot \vec{S}_j$ terms is fixed to ± 2 and also the constant term $\ell(\ell + q)$ is entirely fixed by the spin representation.)

There exist generalizations of the above models both at the classical [38, 39] and at the quantum level [40]. They all share, however, the limitations spelled out above, especially (3). These restrictions are important not only in the quest of more general statistics but also from the more practical point of view of solving spin-chain models with spins not in the fundamental representation, as we will shortly explain. For this reason, yet a different approach will be pursued in the next section, namely the operator approach.

6. Operator approaches

The matrix model connection provided us with a powerful tool that not only allowed us to generalize the notion of identical particle but also led to the full quantum solution of a set of spin-generalized Calogero models.

As noted, however, in the conclusion of the preceding lecture, the matrix model fixes the coefficient of the spin interaction and scalar interaction terms to ± 2 and $\pm\ell(\ell \pm q)$, respectively. We cannot choose these coefficients at will.

We would like to have an approach that defeats this restriction and leads to spin models with arbitrary coupling strengths. (This is necessary to attack spin-chain systems through the infinite-coupling limit trick to be explained later.) Such an approach should also be able to bypass the excursion to matrix models and deal more directly with these systems in an algebraic way. This will be achieved with the exchange operator formalism [11].

6.1. Exchange operator formalism

Consider the operators M_{ij} that permute the *coordinate* degrees of freedom of N particles in one dimension which could, in principle, also have internal degrees of freedom (M for *metathesis*, to avoid confusion with momenta p_i). They satisfy the permutation algebra (symmetric group), in particular

$$M_{ij} = M_{ij}^{-1} = M_{ij}^{\dagger} = M_{ji}, \tag{175}$$

$$[M_{ij}, M_{kl}] = 0, \qquad \text{if} \quad i, j, k, l \text{ distinct}, \tag{176}$$

$$M_{ij}M_{jk} = M_{ik}M_{ij}, \qquad \text{if} \quad i, j, k \text{ distinct}. \tag{177}$$

Any operator A_i on the phase space satisfying

$$M_{ij}A_k = A_k M_{ij}, \qquad \text{if} \quad i, j, k \text{ distinct}, \tag{178}$$

$$M_{ij}A_i = A_j M_{ij} \tag{179}$$

will be called a *one-particle* operator (even though it may involve the coordinates and momenta of many particles).

We construct the following one-particle operators [11]:

$$\pi_i = p_i + \sum_{j\neq i} \mathrm{i}W(x_i - x_j)M_{ij} \equiv p_i + \sum_{j\neq i} \mathrm{i}W_{ij}M_{ij}. \tag{180}$$

We shall view π_i as generalized momenta. To ensure their Hermiticity, the *prepotential* $W(x)$ should satisfy

$$W(-x) = -W(x)^*. \tag{181}$$

We shall construct the corresponding 'free' Hamiltonian from π_i

$$H = \sum_i \frac{1}{2}\pi_i^2. \tag{182}$$

In terms of the original p_i, this Hamiltonian will, in general, contain linear terms. To ensure that such terms are absent, we must further impose

$$W(-x) = -W(x) = \text{real}. \tag{183}$$

With the above restriction, the Hamiltonian H and commutation relations of π_i become

$$[\pi_i, \pi_j] = \sum_k W_{ijk}(M_{ijk} - M_{jik}), \tag{184}$$

$$H = \sum_i \frac{1}{2}p_i^2 + \sum_{i<j} \left(W_{ij}^2 + W'_{ij}M_{ij}\right) + \sum_{i<j<k} W_{ijk}M_{ijk}, \tag{185}$$

where we defined the three-body potential and cyclic permutation as

$$W_{ijk} = W_{ij}W_{jk} + W_{jk}W_{ki} + W_{ki}W_{ij}, \tag{186}$$

$$M_{ijk} = M_{ij}M_{jk}. \tag{187}$$

To obtain an interesting and tractable model, W_{ijk}, which appears in the commutator $[\pi_i, \pi_j]$ and also as a three-body potential, should vanish or at most be a constant. This leads to a functional equation for $W(x)$:

$$W(x)W(y) - W(x+y)[W(x) + W(y)] = \text{const}(=W_{ijk}). \tag{188}$$

We present the solutions:

(a) $W_{ijk} = 0 \to W(x) = \ell/x$.
(b) $W_{ijk} = -\ell^2 < 0 \to W(x) = \ell \cot x$.
(c) $W_{ijk} = +\ell^2 > 0 \to W(x) = \ell \coth x$.

Let us examine each case.

(a) In this case, π_i become

$$\pi_i = p_i + \sum_{j \neq i} \frac{\mathrm{i}\ell}{x_{ij}} M_{ij} \tag{189}$$

and satisfy

$$[\pi_i, \pi_j] = 0. \tag{190}$$

π_i commute, so we can consider them as independent momenta. (They are sometimes referred to as Dunkl operators [41].) The Hamiltonian reads

$$H = \sum_i \frac{1}{2} p_i^2 + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{x_{ij}^2}. \tag{191}$$

We obtain a Calogero-like model with *exchange interactions*. Yet, it is nothing but a free model in the commuting momenta π_i. Integrability is immediate: the permutation-invariant quantities

$$I_n = \sum_i \pi_i^n \tag{192}$$

obviously commute with each other. If we assume that the particles carry no internal degrees of freedom and are bosons or fermions, then $M_{ij} = \pm 1$ on physical states. The model becomes the standard Calogero model and we have proved its integrability in one scoop. (You may be left with a question: the Hamiltonian and the other integrals I_n become the standard Calogero ones if $M_{ij} = \pm 1$, so these reduced integrals will commute on the bosonic or fermionic subspace; but will they also commute on the *full* Hilbert space? Prove for yourself that this is indeed the case.)

We can also construct harmonic oscillator operators [11, 42]. The commutators between x_i and π_i are

$$[x_i, \pi_i] = \mathrm{i}\left(1 + \ell \sum_{j \neq i} M_{ij}\right), \tag{193}$$

$$[x_i, \pi_j] = -\mathrm{i}\ell M_{ij} \qquad (i \neq j). \tag{194}$$

Defining

$$a_i = \tfrac{1}{\sqrt{2}}(\pi_i - \mathrm{i}\omega x_i), \tag{195}$$

$$a_i^\dagger = \tfrac{1}{\sqrt{2}}(\pi_i + \mathrm{i}\omega x_i), \tag{196}$$

we can show that

$$\left[a_i, a_i^\dagger\right] = \omega\left(1 + \ell \sum_{j \neq i} M_{ij}\right), \tag{197}$$

$$\left[a_i, a_j^\dagger\right] = -\omega\ell M_{ij} \qquad (i \neq j), \tag{198}$$

$$[a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0. \tag{199}$$

This is an extended version of the Heisenberg algebra involving the permutation operators. The corresponding oscillator Hamiltonian reads

$$H = \sum_i \frac{1}{2}\left(a_i^\dagger a_i + a_i a_i^\dagger\right) = \sum_i \frac{1}{2}p_i^2 + \sum_i \frac{1}{2}\omega^2 x_i + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{x_{ij}^2} \tag{200}$$

and satisfies

$$[H, a_i] = \omega a_i, \qquad \left[H, a_i^\dagger\right] = \omega a_i^\dagger. \tag{201}$$

This is the harmonic Calogero model with exchange interactions, which becomes again the standard model on bosonic or fermionic subspaces for particles without internal degrees of freedom. Since

$$H = \sum_i a_i^\dagger a_i + \frac{1}{2}N\omega + \frac{1}{2}\ell\omega \sum_{i\neq j} M_{ij}, \tag{202}$$

we see that on bosonic or fermionic spaces the state annihilated by all a_i (if it exists) will be the ground state. Solving $a_i\psi = 0$, we obtain for the ground-state wavefunction

$$\psi_{\mathrm{B}} = \prod_{i<j} |x_{ij}|^\ell \exp\left(-\frac{1}{2}\omega \sum_i x_i^2\right), \tag{203}$$

$$\psi_{\mathrm{F}} = \prod_{i<j} \{\mathrm{sgn}(x_{ij})|x_{ij}|^{-\ell}\} \exp\left(-\frac{1}{2}\omega \sum_i x_i^2\right). \tag{204}$$

For $\ell > 0$, the bosonic state is acceptable, while for $\ell < 0$ the fermionic one is acceptable. In the 'wrong' combinations of statistics and sign of ℓ, the ground state is not annihilated by a_i, but it is still annihilated by all permutation-invariant combinations of a_i.

From (201) we see that we can find the spectrum of this model for fermions or bosons by acting on the ground state with all possible permutation-symmetric homogeneous polynomials in $a_i^\dagger$. A basis for these is, e.g.,

$$A_n = \sum_i \left(a_i^\dagger\right)^n. \tag{205}$$

So the spectrum is identical to non-interacting fermions or bosons, but with a different ground-state energy. For the 'right' combinations of ℓ and statistics, where (204) are the correct ground-state wavefunctions, the ground-state energy is

$$E_0 = \frac{N}{2}\omega + \frac{N(N-1)}{2}|\ell|\omega, \tag{206}$$

which is the correct Calogero result.

Finally, the quantities

$$I_n = \sum_i h_i^n = \sum_i \left(a_i^\dagger a_i\right)^n \tag{207}$$

can be shown to commute [11], and therefore this system is also integrable. It is left as an exercise to find the commutation relations of h_i and show that $[I_n, I_m] = 0$.

(b) In the case $W(x) = \ell \cot x$, we have

$$\pi_i = p_i + \mathrm{i}\cot x_{ij} M_{ij}, \tag{208}$$

$$[\pi_i, \pi_j] = -\ell^2 \sum_k (M_{ijk} - M_{jik}), \tag{209}$$

so the momenta are now coupled. The Hamiltonian becomes

$$H = \sum_i \frac{1}{2} p_i^2 + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{\sin^2 x_{ij}} - \ell^2 \left(\frac{N(N-1)}{2} + \sum_{i<j<k} M_{ijk} \right). \tag{210}$$

We obtain the Sutherland model with exchange interactions plus an extra term. On bosonic or fermionic states, this becomes an overall constant and we recover the standard Sutherland model. Again, since H is by construction positive definite, if a state satisfying $\pi_i \psi = 0$ exists it will be the ground state. We obtain

$$\psi_{\mathrm{B}} = \prod_{i<j} |\sin x_{ij}|^{\ell}, \tag{211}$$

$$\psi_{\mathrm{F}} = \prod_{i<j} \mathrm{sgn}(x_{ij}) |\sin x_{ij}|^{\ell}, \tag{212}$$

which are acceptable for the same combinations of ℓ and statistics as before. For both cases $M_{ijk} = 1$, so

$$E_0 = \ell^2 \frac{N(N^2-1)}{24} \tag{213}$$

is the correct Sutherland model ground-state energy. The excited states can again be obtained in a (rather complicated) algebraic way [43]. Finally, the quantities

$$\tilde{\pi}_i = \pi_i + \ell \sum_{j \neq i} M_{ij} = p_i + \mathrm{e}^{\mathrm{i}x_i} \sum_{j \neq i} \frac{2\ell}{\mathrm{e}^{\mathrm{i}x_i} - \mathrm{e}^{\mathrm{i}x_j}} M_{ij} \tag{214}$$

can be shown to have the same commutation relations as h_i defined previously for the harmonic system. Therefore, the integrals constructed from them

$$I_n = \sum_i \tilde{\pi}_i^n \tag{215}$$

commute and the model is integrable.

(c) For $W(x) = \ell \coth x$, we have a similar commutation relation and a Hamiltonian

$$H = \sum_i \frac{1}{2} p_i^2 + \sum_{i<j} \frac{\ell(\ell - M_{ij})}{\sinh^2 x_{ij}} + \ell^2 \left(\frac{N(N-1)}{2} + \sum_{i<j<k} M_{ijk} \right). \tag{216}$$

This is the inverse-hyperbolic-sine-square model and supports only scattering states. Its integrability can be obtained as for the Sutherland model above, or simply as an 'analytic continuation' of that model for imaginary period of space. We will not examine it any further.

In conclusion, an exchange family of models was introduced, solved and related to the standard Calogero models in spaces of definite symmetry. It is remarkable that all these proofs work directly, and only, at the quantum domain (there is no classical analogue of M_{ij}).

6.2. Systems with internal degrees of freedom

We can easily extend the previous results for particles with internal degrees of freedom. For this, assume that the particles are *distinguishable* or, equivalently, that they carry a number q of (discrete) internal degrees of freedom (species) that can be used to (partially) distinguish them. Their states are spanned by $|x, \sigma\rangle$, where $\sigma = 1, \ldots, q$ counts internal states. The total permutation operator T_{ij}, then, is

$$T_{ij} = M_{ij}\sigma_{ij}, \tag{217}$$

where σ_{ij} is the operator that permutes the internal states of particles i and j.

Let us, then, simply take states that are bosonic or fermionic under total particle exchange: $T_{ij} = \pm 1$. On such states

$$M_{ij} = \pm\sigma_{ij} \tag{218}$$

and the Calogero and Sutherland exchange model Hamiltonians become [44]

$$H_{\rm c} = \sum_i \frac{1}{2}p_i^2 + \sum_i \frac{1}{2}\omega^2 x_i + \sum_{i<j} \frac{\ell(\ell \mp \sigma_{ij})}{x_{ij}^2}, \tag{219}$$

$$H_{\rm s} = \sum_i \frac{1}{2}p_i^2 + \sum_{i<j} \frac{\ell(\ell \mp \sigma_{ij})}{\sin^2 x_{ij}} - \ell^2\left(\frac{N(N-1)}{2} + \sum_{i<j<k}\sigma_{ijk}\right). \tag{220}$$

We get the Calogero and Sutherland models with spin-exchange interactions. From the completeness relation for the fundamental $SU(q)$ generators T^a

$$\sum_{a=1}^{q^2-1} T^a_{\alpha\beta}T^a_{\gamma\delta} = \frac{1}{2}\delta_{\alpha\delta}\delta_{\gamma\beta} - \frac{1}{2q}\delta_{\alpha\beta}\delta_{\gamma\delta}, \tag{221}$$

we deduce the form of the operators σ_{ij}

$$\sigma_{ij} = 2\vec{S}_i \cdot \vec{S}_j + \frac{1}{q}, \tag{222}$$

where S_i^a acts as T^a on the internal states of particle i. So the spin-dependent interaction coefficient of the potential in the Hamiltonian takes the form [44–47]

$$\mp\ell\left(2\vec{S}_i \cdot \vec{S}_j \mp \ell + \frac{1}{q}\right). \tag{223}$$

We have recovered the ferromagnetic and antiferromagnetic spin model of the previous section but with *arbitrary* coefficient! On the other hand, the spins are necessarily in the fundamental of $SU(q)$. So we have obtained a generalization of the coupling constant with respect to the matrix model but a restriction of the allowed spins.

Note that ℓ here is an arbitrary parameter, while ℓ in (171) was the size of the symmetric representation of S_i. For $\ell = 1$ and spins in the fundamental, the matrix model and the exchange operator model agree. It is interesting to note that we can go from ferromagnetic to antiferromagnetic interactions either by changing the sign of ℓ or by changing the statistics of the particles.

The solution of the above models can be obtained algebraically. For the spin-Sutherland model, this is rather complicated and is related to the so-called Yangian symmetry [48–51]. For the spin-Calogero model, it is easier [52]. Let us concentrate on the model with interaction $\ell\big(-2\vec{S}_i \cdot \vec{S}_j \mp \ell - \frac{1}{q}\big)$ and define the operators

$$A_n^\dagger = \sum_i \big(a_i^\dagger\big)^n, \qquad \big(A_n^a\big)^\dagger = \sum_i \big(a_i^\dagger\big)^n S_i^a \tag{224}$$

and their Hermitian conjugates. These form a complete set for all permutation-symmetric creation and annihilation operators for all species of particles. Yet, the commutators among themselves and with H do not involve ℓ. They create, therefore, the same spectrum of excitations over the ground state as N non-interacting bosons or fermions with q species. For $\ell > 0$, the ground state is the bosonic one:

$$\psi_{\rm B} = \prod_{i<j} |x_{ij}|^{\ell} \exp\left(-\frac{1}{2}\omega \sum_i x_i^2\right) \chi_{\rm s}(\{\sigma_i\}), \tag{225}$$

where $\chi_{\rm s}$ is a totally symmetric state in σ_i. The set of all $\chi_{\rm s}$ forms the N-fold symmetric irrep of the total spin $S = \sum_i S_i$. Therefore, the ground state is $(N+q-1)!/N!(q-1)!$ times degenerate. For $\ell < 0$, the above is not normalizable anymore. But we remember that we can obtain the same model by starting from fermions and the opposite coupling $-\ell > 0$. The ground state, then, is of a fermionic type

$$\psi_{\rm F} = \sum_P (-1)^P \left(\prod_i \delta_{\sigma_i,\alpha_i}\right) \prod_{i<j} |x_{ij}|^{-\ell} x_{ij}^{\delta_{\alpha_i,\alpha_j}} \exp\left(\frac{1}{2}\omega \sum_i x_i^2\right), \tag{226}$$

where P are total particle permutations and α_i are a set of fixed values for the indices σ_i that determine the state. Clearly, the ground state will be obtained for the minimal total power of x_i appearing above, and that will happen for a maximally different set of values α_i. These states form the n-fold antisymmetric irrep of the total spin S, where $n = N \pmod q$. The ground state is, thus, $q!/n!(q-n)!$ times degenerate. The above spectra will come handy later.

6.3. Asymptotic Bethe ansatz approach

We already mentioned that there are elaborate algebraic approaches to derive the spectrum of the spin-Sutherland model, based on the Yangian symmetry. We will, instead, take a lower key approach which reproduces the same spectra and is physically more lucid, although not as rigorous. We will take the ABA route.

Consider *distinguishable* particles of the exchange-Calogero type without external potential, coming in with asymptotic momenta k_i and scattering off each other. Before scattering, their positions are in some definite ordering determined by the ordering of their momenta (it is the inverse of that ordering).

The key observation is that, after scattering, the particles have simply 'gone through' each other with *no* backscattering [53]. The impenetrable $1/x^2$ potential has become completely penetrable in the presence of the exchange term! You can prove this fact by examining the asymptotic properties of a simultaneous eigenstate of $\pi_1, \ldots, \pi_N$ which is obviously an energy eigenstate: at $x_i \to \pm\infty$, the prepotential terms are vanishing and we simply have eigenstates of the individual p_i. Since there are *no* pieces with the values of p_i permuted (coming from backscattering), we have complete transmission. (To explicitly see how it works, it is instructive to consider the two-body problem, decompose it into symmetric and antisymmetric parts, scatter and recombine the parts after scattering. A relative phase of π between the two parts is what produces the effect.)

(Puzzle: what happens with the correspondence principle? With $\hbar$ back in, the interaction coefficient is $\ell(\ell - \hbar M_{ij})$. How can a term of order $\hbar$ produce such a dramatic effect, particles going through each other, in the $\hbar \to 0$ limit?)

So the only effect of the scattering is a phase shift of the wavefunction which, as we have said, is the sum of two-body phases

$$\theta_{\rm sc} = \frac{N(N-1)}{2}\pi\ell. \tag{227}$$

This is true on an infinite space. On a periodic space, we can still use the above result, together with the requirement for periodicity for the wavefunction, to derive the spectrum. This is the ABA method and is expected to reproduce the correct results in the thermodynamic limit of many particles at constant density [53]. It gives, in fact, the *exact* answer for the Sutherland model [2], so we can expect it to also work in the present case. For a space of period 2π, the result is

$$2\pi k_i + \sum_{\pi} \ell \,\mathrm{sgn}(k_i - k_j) = 2\pi n_i. \tag{228}$$

The left-hand side counts the total phase picked up by a particle going round the space and scattering off to the other particles in the way. n_i are arbitrary integers, ensuring periodicity. There are, however, some constraints on the choice of n_i that are imposed by continuity from the $\ell = 0$ case:

- If $k_i \leqslant k_j$, then $n_i \leqslant n_j$.
- If $n_i = n_j$, there is a *unique* solution, that is, $k_i < k_j$ and $k_i > k_j$ represent the same state.

These rules are important to avoid overcounting and to discard spurious solutions. With these, the spectrum obtained is the same as that derived with more rigorous methods. For the ordering $n_1 \leqslant \cdots \leqslant n_N$, the solution for k_i is

$$k_i = n_i + \ell \left(i - \frac{N+1}{2} \right) \tag{229}$$

and similarly for other orderings. We see that the ABA momenta k_i are the same as the pseudomomenta that we have previously defined.

The bottom line is that the spectrum and degeneracies are the same as those of distinguishable particles obeying generalized selection rules for their momentum. Still, what fixes the degeneracy of states is the different ways that we can distribute the particles to the quantum numbers n_i, rather than k_i (see, especially, the second rule above). A state of N particles with the same n_i, for instance, is nondegenerate although they, seemingly, have different k_i which would imply a permutation degeneracy.

For particles with spin, the above construction, in combination with the trick of the previous subsection of starting with fermions or bosons, produces a spectrum with degeneracies, the same as those of free particles (n_i are 'free' quantum numbers). As argued before, for ferromagnetic interactions we must choose bosons and combine their spins accordingly, while for antiferromagnetic interactions we must choose fermions. To spell it out, this means the following:

(1) Choose a set of quantum numbers n_i. The ordering is immaterial, since we have identical particles, so you can choose $n_1 \leqslant \cdots \leqslant n_N$.
(2) Place your particles on these quantum numbers and put their spins in the appropriate state. For the ferromagnetic case, treat them as bosons: the total spin of particles with the same n_i transforms in the symmetric tensor product of their spins. For the antiferromagnetic case, treat them as fermions: the total spin of particles with the same n_i transforms in the antisymmetric tensor product of their spins; clearly up to q can have same n_i in this case.
(3) Calculate the energy of this state in terms of the ABA momenta (229): $E = \sum_i k_i^2$.

It should be obvious that similar rules applied to the spin-Calogero system reproduce the spectrum derived in the last subsection. This method can be used to calculate both the statistical mechanics (large N) of these systems and the few-body spectra.

6.4. The freezing trick and spin models

Now that we have a tractable way of solving spin-Calogero systems with arbitrary strength of interaction we can introduce the freezing trick [55] and deal with spin-chain models.

Consider, first, the previous ferromagnetic or antiferromagnetic spin-Sutherland model. Take the limit $\ell \to \infty$. The potential between the particles goes to infinity, so for any finite-energy state the particles will be nearly 'frozen' to their classical equilibrium positions. In fact, even the excitation energies around that configuration will go to infinity: the ground-state energy scales like ℓ^2, while the excitations scale like $N\ell n + n^2$ with n some excitation parameter. So, to leading order in ℓ the spectrum becomes linear and of order ℓ. These excitations correspond, essentially, to phonon modes of small oscillations around the equilibrium positions of particles. The 'stiffness' of oscillations is, of course, proportional to the strength of the potential ℓ^2 and the spectrum is proportional to the frequency, of order ℓ.

The quantum fluctuations of the particle positions in any state will scale like the inverse square root of the oscillator frequency, that is, like $1/\sqrt{\ell}$. But, in the Hamiltonian, the piece coupling the spins to the kinematical degrees of freedom is proportional to $1/\sin^2 x_{ij}$. In the large-ℓ limit, thus, this term becomes a constant equal to its classical equilibrium value; so, in that limit, spin and kinematical degrees of freedom decouple. (Note that the spin part is also of order ℓ.) The Hamiltonian becomes

$$H = H_{\rm S} + \ell H_{\rm spin}, \tag{230}$$

with $H_{\rm S}$ the spinless Sutherland Hamiltonian and $H_{\rm spin}$ the spin part

$$H_{\rm spin} = \mp \sum_{i<j} \frac{2\vec{S}_i \cdot \vec{S}_j}{4\sin^2 \frac{\bar{x}_{ij}}{2}}, \tag{231}$$

where the classical equilibrium positions $\bar{x}_j$ are equidistant points on the circle:

$$\bar{x}_j = \frac{2\pi j}{N}. \tag{232}$$

The Hamiltonian (231) describes a spin chain consisting of a regular periodic lattice of spins in the fundamental of $SU(q)$ coupled through mutual ferromagnetic or antiferromagnetic interactions of strength inversely proportional to their chord distance. It is the well-known $SU(q)$ Haldane–Shastry (HS) model [8, 9]. According to the above, its spectrum can be found by taking the full spectrum of the corresponding spin-Sutherland model in the large-ℓ limit, 'modding out' the spectrum of the spinless model and rescaling by a factor $1/\ell$. Each state will inherit the spin representation of its 'parent' spin-Sutherland state. So, both the energy and the total spin of the states of the HS model can be determined this way. Commuting integrals of this model [10] can also be obtained this way [54]. At the level of the partition function at some temperature T, we have

$$Z_{\rm spin}(T) = \lim_{\ell\to\infty} \frac{Z(\ell T)}{Z_{\rm S}(\ell T)}. \tag{233}$$

From this, the thermodynamics of the spin-chain model can be extracted.

We will not give the details of this construction here. We urge anyone interested to solve this way a few-site (two or three) spin chain see how it works and deduce the 'construction rules' for the spectrum of a general spin chain. Let us simply state that the many degeneracies of the spectrum of the HS model (larger than the total spin $SU(q)$ symmetry would imply), which is algebraically explained by the existence of the Yangian symmetry, can, in this approach, be explained in terms of the degeneracies of free particles. (The degeneracies are not *identical*, due to the modding procedure, but related.)

For the spin-Calogero model, a similar limit can be taken, scaling also the external oscillator frequency as $\omega \to \ell\omega$ to keep the system bound. The classical equilibrium positions of this model are at the roots of the Nth Hermite polynomial. We obtain, therefore, a non-regular lattice of spins interacting with a strength inversely proportional to the square of their distance [55]. The spectrum of this model can be found quite easily with the above method. Again, we refer to the literature for details [52, 56, 57].

In the continuum limit ($N \to \infty$), the antiferromagnetic version of both the above models becomes $c = 1$ conformal field theories, the HS containing both chiral sectors while the inhomogeneous harmonic one containing just one sector.

Other models exist and can be solved in this spirit: hierarchical (many-coupling) models [58], supersymmetric models [59, 60], 'twisted' models [61], etc. All, however, work only for the fundamental representation of some internal group. The big, important open problem is to crack a particle system with a *higher* representation for the spins and *arbitrary* coupling strength. If this is done, through the freezing trick we will be able to solve a spin chain with spins in a higher representation. This is interesting since we could then see if the antiferromagnetic system for integer $SU(2)$ spins develops a mass gap, according to the Haldane conjecture [62].

7. Folding the Calogero model: new systems by reduction

In previous sections, we presented mainly two variants of the Calogero model: the rational (Calogero proper) and the trigonometric (Sutherland) one. There are other possibilities, in which the two-body potential is further generalized to a Weierstrass function (and its hyperbolic reduction) as well as systems with 'reflection' symmetry. Spin-Calogero models along these generalizations and involving spin 'twists' are also possible.

Rather than giving an independent analysis of these models, we prefer to present how they can be obtained as appropriate reductions of standard (spin) Calogero models. The advantage is a conceptual unification of the various models and the realization that the 'root' model of particles with inverse-square interaction contains all the fundamental information about these systems, everything else stemming from it as particular sectors and reductions.

7.1. Reduction of spinless models

The main idea is to reduce the Calogero model by some of its discrete symmetries, akin to the 'orbifold' construction in field and string theory.

Consider any Hamiltonian dynamical system with some discrete symmetries D. Its equations of motion remain invariant under the phase space mapping $\phi \to D(\phi)$, where ϕ are phase space variables. Then the reduction to the invariant subspace $\phi = D(\phi)$ is kinematically preserved; that is, the equations of motion do not move the system out of this subspace. Therefore, reducing the initial value data to this subspace trivially produces a system as solvable as the original one. The motion will be generated by the original Hamiltonian on the reduced space.

The starting point will be the plain vanilla scattering Calogero model

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{g}{x_{ij}^2}, \tag{234}$$

where $x_{ij} = x_i - x_j$. The considered symmetries are:

- Translation invariance T: $x_i \to x_i + \mathbf{a}$, $p_i \to p_i$.

- Parity P: $x_i \to -x_i$, $p_i \to -p_i$.
- Permutation symmetry M: $x_i \to x_{M(i)}$, $p_i \to p_{M(i)}$ with M any element of the permutation group of N particles.

Other symmetries will not be useful for our purposes.

A direct reduction of the system by any of the above symmetries does not produce anything nontrivial or sensible: $\phi = T(\phi)$ is possible only in the trivial case $a = 0$, while $\phi = P(\phi)$ and $\phi = M(\phi)$ require (some) of the particle coordinates to coincide, which is excluded by the infinite two-body potential. We get useful systems only when reducing through appropriate products of the above symmetries. These are as follows:

(a) $D = PM$: we reduce by P and a particular permutation: $M(i) = N - i + 1$ (or any other in the same conjugacy class). M is uniquely fixed from the fact that $D^2(\phi) = M^2(\phi)$ is a pure permutation and for it not to make any two different particle coordinates to coincide we must have $M^2 = 1$. Further, if for any particle $M(i) = i$, then $D(x_i) = -x_i$ and the corresponding coordinate is set to zero. So $M(i) = i$ can happen for at most one i (so that no two or more particle coordinates are put to zero). So M must be a collection of rank-2 (two-body) permutations, and possibly a rank-1 (trivial) permutation if the number of particles is odd, which can always be represented in the above form.

The constraint

$$x_i = -x_{N-i+1}, \qquad p_i = -p_{N-i+1} \tag{235}$$

effectively reduces the original system into two mirror images. The reduced Hamiltonian is

$$\begin{aligned} H &= \sum_{i=1}^{N'} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{g}{(x_i - x_j)^2} + \frac{1}{2} \sum_{i \neq j} \frac{g}{(x_i + x_j)^2} + \sum_i \frac{g'}{x_i^2}, \\ N' &= \left[\frac{N}{2} \right], \qquad g' = g \left(\frac{1}{4} + 2 \left\{ \frac{N}{2} \right\} \right), \end{aligned} \tag{236}$$

where an overall factor of 2 has been discarded and $[\cdot]$, $\{\cdot\}$ denote integer and fractional parts, respectively. The second term in the potential is the interaction of each particle with the mirror image of each other particle; the third part accounts for the interaction of each particle with the mirror image of itself, and with a particle fixed at the origin by the constraint (for odd N).

Parity symmetry persists in the case where an external harmonic oscillator potential is added to the system, promoting it to the confining, rather than scattering, Calogero model:

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{g}{x_{ij}^2} + \sum_{i=1}^{N} \frac{1}{2} \omega^2 x_i^2. \tag{237}$$

Reduction by $D = PM$ produces an integrable system similar as above with the added harmonic oscillator potential.

(b) $D = TM$: no finite-rank element of the permutation group will do, since repeated application of D would eventually lead to $x_i = x_i + m\mathbf{a}$. We overcome this by starting with N' particles and taking the limit $N' \to \infty$. We pick the element $M(i) = i + N$ for some finite N which, for infinite N', is of infinite rank. The constraint

$$x_{i+N} = x_i + \mathbf{a}, \qquad p_{i+N} = p_i \tag{238}$$

leads to a system consisting of infinitely many copies of a finite system displaced by multiples of $\mathbf{a}$. We can parametrize the particle indices by the pair (i, m), $i = 1, \ldots, N$ and $m \in \mathbf{Z}$, where the original index is $i + mN$. The constraint now reads

$$x_{i,m} = x_i + m\mathbf{a}, \qquad p_{i,m} = p_i. \tag{239}$$

The resulting system is infinite copies of an N-body system. The reduced Hamiltonian is

$$H = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{m,n=-\infty}^{\infty} \sum_{i,j} \frac{g}{(x_{ij} + m\mathbf{a} - n\mathbf{a})^2}$$
$$= \frac{1}{2} \sum_{m'=-\infty}^{\infty} \left\{ \sum_{i=1}^{N} p_i^2 + \sum_{m=-\infty}^{\infty} \sum_{i \neq j} \frac{g}{(x_{ij} + ma)^2} + \sum_{i} \sum_{m \neq 0} \frac{g}{(ma)^2} \right\}. \quad (240)$$

In the above summation, terms with $i = j, m = n$ are omitted since they correspond to self-interactions of particles that are excluded from the original Calogero model. The summation over m' above accounts for the infinite periodically repeating copies of the system and can be dropped. The infinite m-summation accounts for the interaction of each particle with the multiple images of each other particle and can be performed explicitly. We eventually get

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{g\pi^2}{\mathbf{a}^2 \sin^2 \pi \frac{x_{ij}}{\mathbf{a}}}, \quad (241)$$

that is, the Sutherland model. In the above, we omitted an irrelevant constant term equal to $gN\frac{\pi^2}{6\mathbf{a}^2}$, arising from the last term in the summand corresponding to particle terms with $i = j$, which account for the interaction of each particle with its own infinite images.

(c) We can formally extend the T symmetry to complex parameter $\mathbf{a}$. As long as there is a subset of coordinates in the reduced phase space that remains real and generates all other coordinates through use of D, we will have a well-defined real subsystem. Applying $D = TM$ for infinitely many particles parametrized by a double index $(i, j), i, j \in \mathbf{Z}$, for two complex translations $\mathbf{a}$ and $\mathbf{b}$, the constraint is

$$x_{i,j} = x_{i+N',j} + \mathbf{a} = x_{i,j+M'} + \mathbf{b}, \qquad p_{i,j} = p_{i+N',j} = p_{i,j+M'} \quad (242)$$

and we end up with a finite system with $N = M'N'$ particles periodically repeating on the complex plane. (Clearly, the specific separation of N into N' and M' is irrelevant; we could take, e.g., $N' = N, M' = 1$.) Similarly to the Sutherland case, the Hamiltonian becomes infinitely many copies of

$$H = \sum_{i=1}^{N'} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i,j} \sum_{m,n=-\infty}^{\infty} \frac{g}{(x_{ij} + m\mathbf{a} + n\mathbf{b})^2}. \quad (243)$$

The above sum has a logarithmic ambiguity that is easily regulated by subtracting the constant $g/(m\mathbf{a} + n\mathbf{b})^2$ from each term. We end up with

$$H = \sum_{i=1}^{N'} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} g\mathcal{P}(x_{ij}|\mathbf{a}, \mathbf{b}), \quad (244)$$

that is, a model with an elliptic Weierstrass potential.

(d) $D_1 = PM$ together with $D_2 = TM$: this is a combination of (a) and (b). We work again with an infinite number of particles. We impose two constraints:

$$\begin{aligned} x_{-j+1-\epsilon} &= -x_j, & p_{-j+1-\epsilon} &= -p_j, \\ x_{j+N'} &= x_j + \mathbf{a}, & p_{j+N'} &= p_j, \end{aligned} \quad (245)$$

where $\epsilon = 0, 1$ (any other choice of ϵ is equivalent to one of these). Parametrizing $j = i + mN'$ by the pair $i, m, i = 1, \ldots, N', m \in \mathbf{Z}$, we have

$$\begin{aligned} x_{N-i+1-\epsilon} &= \mathbf{a} - x_{i,m}, & p_{N-i+1-\epsilon} &= -p_i, \\ x_{i,m} &= x_i + m\mathbf{a}, & p_{i,m} &= p_i, \end{aligned} \quad (246)$$

so we end up with a finite system of $N = [(N' - \epsilon)/2]$ particles. The reduced Hamiltonian is infinitely many copies of

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{g\pi^2}{\mathbf{a}^2 \sin^2 \pi \frac{x_i - x_j}{\mathbf{a}}} + \frac{1}{2} \sum_{i \neq j} \frac{g\pi^2}{\mathbf{a}^2 \sin^2 \pi \frac{x_i + x_j}{\mathbf{a}}} + \frac{1}{2} \sum_{i} \frac{g'\pi^2}{\mathbf{a}^2 \sin^2 \pi \frac{2x_i}{\mathbf{a}}} + \frac{1}{2} \sum_{i} \frac{g''\pi^2}{\mathbf{a}^2 \sin^2 \pi \frac{x_i}{\mathbf{a}}}, \tag{247}$$

with

$$g' = g\left(\frac{1}{2} + 8\left\{\frac{N - \epsilon}{2}\right\}\right), \qquad g'' = g\left(\epsilon - 2\left\{\frac{N - \epsilon}{2}\right\}\right). \tag{248}$$

The coordinates x_i can all be taken in the interval $(0, \mathbf{a}/2)$ and the particles interact with their infinite mirror images with respect to mirrors placed at $x = 0$ and $x = \mathbf{a}/2$ and with particles fixed at $x = 0$ (if $\epsilon = 1$) and at $x = \mathbf{a}/2$ (if $2\{(N - \epsilon)/2\} = 1$). A similar construction can be performed with two complex translations, as in (c) plus one parity reversal. We obtain a similar model but with elliptic functions appearing instead of inverse sine squares.

7.2. Integrals of motion via reduction

The above exhausts the possibilities for spinless particles. Before we proceed to the more interesting case of particles with spin, it is instructive to demonstrate how the above construction reproduces the conserved integrals of motion of the reduced system. We will consider case (b) as the most generic, case (a) being rather trivial.

The Lax matrix of the original scattering Calogero model, as given in an earlier section, is

$$L_{ij} = p_i \delta_{ij} + (1 - \delta_{ij}) \frac{\mathrm{i}\ell}{x_{ij}}. \tag{249}$$

Traces of powers of L produce the integrals of motion in involution for the model:

$$I_k = \operatorname{tr} L^k, \qquad k = 1, \ldots, N. \tag{250}$$

For the system of case (b), we promote the index i into a pair (i, m) and choose $x_{i,m} = x_i + m\mathbf{a}$ as in (239). The resulting infinite-dimensional matrix $L_{im,jn}$ can be thought of as consisting of infinitely many blocks of size $N \times N$, m, n labelling the blocks and i, j the elements of each block:

$$L_{im,jn} = p_i \delta_{mn} \delta_{ij} + (1 - \delta_{mn}\delta_{ij}) \frac{\mathrm{i}\ell}{x_{ij} + (m - n)\mathbf{a}}. \tag{251}$$

We observe a block 'translational invariance' of the matrix L in the indices m, n, which reflects the invariance of the model under a translation by $\mathbf{a}$. Due to this, we can trade the pair m, n for a single index $m - n$

$$L_{im,jn} \equiv L_{m-n;ij} \tag{252}$$

and thus L becomes an infinite collection of $N \times N$ matrices L_n labelled by n. We define the Fourier transform $L(\sigma)$:

$$L(\sigma) = \sum_n \mathrm{e}^{\mathrm{i}n\sigma} L_n \tag{253}$$

in terms of which L_n is

$$L_n = \frac{1}{2\pi} \int_0^{2\pi} L(\sigma)\, \mathrm{e}^{-\mathrm{i}n\sigma}. \tag{254}$$

The corresponding integrals of motion I_k are traces of powers of L. Denoting by Tr the trace in the infinite-dimensional space labelled by i, m and by tr the trace in the N-dimensional space labelled by i alone, we have

$$\begin{aligned} I_k &= \operatorname{Tr} L^k = \sum_{n_1,\dots,n_k} \operatorname{tr}\bigl(L_{n_1-n_2}\cdots L_{n_k-n_1}\bigr) \\ (m_i \equiv n_i - n_{i+1}) &= \sum_{n_1} \sum_{m_1,\dots,m_{k-1}} \operatorname{tr}\bigl(L_{m_1}\cdots L_{m_{k-1}} L_{-m_1\cdots-m_{k-1}}\bigr). \end{aligned} \tag{255}$$

The sum over n_1 above produces a trivial infinity. This is due to the summation over the infinite copies of the system, just as in the case of the Hamiltonian, and will be dropped. In terms of the Fourier transformed $L(\sigma)$, the reduced I_k simply become

$$I_k = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr} L(\sigma)^k \,\mathrm{d}\sigma. \tag{256}$$

It is now a matter of calculating $L(\sigma)$. From the Fourier transform

$$\sum_n \frac{\mathrm{e}^{\mathrm{i}n\sigma}}{n+x} = \frac{\pi\,\mathrm{e}^{\mathrm{i}(\pi-\sigma)x}}{\sin\pi a} \qquad \text{for} \quad 0<\sigma<2\pi, \tag{257}$$

we obtain for $L(\sigma)$

$$L(\sigma)_{ij} = \mathrm{e}^{\mathrm{i}(\pi-\sigma)x_{ij}}\left[p_i\delta_{ij} + (1-\delta_{ij})\frac{\mathrm{i}\pi\ell}{\mathbf{a}\sin\pi\frac{x_{ij}}{\mathbf{a}}} - \ell\frac{\pi-\sigma}{\mathbf{a}}\delta_{ij}\right], \tag{258}$$

where the last diagonal term linear in $\pi-\sigma$ came from terms with $i=j, n\neq 0$ in $L_{n,ij}$. The matrix inside the square brackets, apart from this linear part, is the Lax matrix $\tilde{L}$ of the Sutherland model:

$$\tilde{L}_{ij} = p_i\delta_{ij} + (1-\delta_{ij})\frac{\mathrm{i}\pi\ell}{\mathbf{a}\sin\pi\frac{x_{ij}}{\mathbf{a}}}. \tag{259}$$

Substituting (258) in (256) we note that the exponential factors cancel (due to $x_{i_1i_2}+\cdots+x_{i_{k-1}i_1}=0$) and we are left with

$$\begin{aligned} I_k &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\left(\tilde{L} - \ell\frac{\pi-\sigma}{\mathbf{a}}\right)^k \mathrm{d}\sigma = \sum_{s=0}^{k} \operatorname{tr}\frac{k!}{s!(k-s)!}\tilde{L}^{k-s}\frac{(-\ell)^s}{2\pi\mathbf{a}^s}\int_0^{2\pi}(\pi-\sigma)^s\,\mathrm{d}\sigma \\ &= \sum_{n=0}^{[k/2]} \frac{k!}{(2n+1)!(k-2n)!}\left(\frac{\pi\ell}{\mathbf{a}}\right)^{2n}\tilde{I}_{k-2n}, \end{aligned} \tag{260}$$

where $\tilde{I}_k = \operatorname{tr}\tilde{L}^k$ are the conserved integrals of the Sutherland model. We obtain a linear combination of the integral $\tilde{I}_k$ and lower integrals of the same parity. The appearance of the lower integrals originates from the interaction of each particle with its own infinite images. We saw an example of such a term in the constant potential term that we omitted from the reduced Hamiltonian of case (b). In conclusion, we have recovered the integrals of the Sutherland model.

7.3. Reduction of spin-Calogero systems

We extend now these considerations to systems of particles with internal classical $U(n)$ degrees of freedom. The corresponding starting spin-Calogero system can be obtained by nontrivial reductions of the Hermitian matrix model, analogous to those that led to the spin-Sutherland model from the unitary matrix model. Alternatively, we may obtain the spin-Calogero model

by taking the infinite-volume limit of the periodic spin-Sutherland model. We shall also consider the classical version of the model, in which the spins become classical phase space variables whose Poisson brackets generate the $U(n)$ algebra.

The Hamiltonian of this model reads

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{\mathrm{tr}(S_i S_j)}{x_{ij}^2}. \tag{261}$$

S_i are a set of independent classical $U(n)$ spins of rank 1 and length ℓ, that is, $n \times n$ rank-1 Hermitian matrices satisfying

$$\mathrm{tr}(S_i)^2 = \ell^2 \tag{262}$$

and with Poisson brackets

$$\{(S_i)_{ab}, (S_j)_{cd}\} = -\mathrm{i}\delta_{ij} \left[(S_i)_{ad}\delta_{cb} - \delta_{ad}(S_i)_{cb}\right]. \tag{263}$$

This can be thought of as the classical limit of the spin models obtained in the quantum treatment of the matrix model. The spin representation was restricted to be totally symmetric, which classically translates to the rank-1 condition.

Just as in the quantum case, such spins can be realized in terms of classical oscillators:

$$(S_i)_{ab} = \bar{A}_i^a A_i^b, \qquad a, b = 1, \ldots, n, \tag{264}$$

where $\left(A_i^a, \bar{A}_i^a\right)$ are a set of nN independent classical harmonic oscillator canonical pairs with Poisson brackets:

$$\left\{A_i^a, \bar{A}_j^b\right\} = \mathrm{i}\delta_{ij}\delta_{ab} \tag{265}$$

and satisfying the constraint

$$\sum_a \bar{A}_i^a A_i^a = \ell \qquad \text{for all } i. \tag{266}$$

The above model, in addition to the previous symmetries T, P and M, also possesses the symmetry.

- Spin rotations U: $S_i \to U S_i U^{-1}$, with U a constant unitary $n \times n$ matrix. (x_i, p_i) remain unchanged.

Again, reduction by this symmetry alone leads to no interesting system (implying either $U = 1$ or $S_i = 0$). Reduction by PUM or TUM, however, much along the lines of the previous PM and TM reductions, produces new and nontrivial results.

(e) $D = PUM$ with P and M as in (a) and U a unitary matrix satisfying $U^2 = 1$ (this is necessary since P and M are of rank 2). The constraints are

$$x_i = -x_{N-i+1}, \qquad p_i = -p_{N-i+1}, \qquad S_i = U S_{N+i-1} U^{-1}. \tag{267}$$

The reduced Hamiltonian acquires the form

$$H = \sum_{i=1}^{N'} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{\mathrm{tr}(S_i S_j)}{(x_i - x_j)^2} + \frac{1}{2} \sum_{i \neq j} \frac{\mathrm{tr}(S_i U S_j U^{-1})}{(x_i + x_j)^2} + \sum_i \frac{\mathrm{tr}(S_i U S_i U^{-1})}{4x_i^2} + g \sum_i \frac{\mathrm{tr}(S_i S_0)}{x_i^2}, \tag{268}$$

where $N' = [N/2]$, $g = 2\{N/2\}$ and $S_0 = U S_0 U^{-1}$ is an extra spin degree of freedom.

The form of the Hamiltonian for the reduced model and its physical interpretation simplifies with an appropriate choice of basis for the spins: by using the U-invariance of

the full model, we can perform a unitary rotation V to all spins $S_i \to V S_i V^{-1}$. This transforms the matrix U appearing in (269) into $U \to V^{-1}UV$. With an appropriate choice of V we can always choose U to be diagonal: $U = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi_a})$. Because of the constraint $U^2 = 1$, this means that U can take the form $U = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ with n_1 (n_2) entries equal to 1 (resp. -1). So we see that the original $U(n)$-invariance of the model has been broken to $U(n_1) \times U(n_2)$. If $n_1 = n_2$, there is an additional Z_2 exchange symmetry.

As in the spinless case (a), we could have started with a spin-Calogero model in an external oscillator potential (which shares the same U and P symmetries) and obtain a model as above with the extra confining harmonic potential.

(f) $D = TUM$ with T and M as in (b) and U any unitary matrix. The constraint on the phase space is

$$x_i = x_{i+N} + \mathbf{a}, \qquad p_i = p_{i+N}, \qquad S_{i+N} = U S_i U^{-1}. \tag{269}$$

The system becomes, again, infinite copies of a-translated and U-rotated systems, and the reduced Hamiltonian is

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i,j} \sum_{m=-\infty}^{\infty} \frac{\mathrm{tr}(S_i U^m S_j U^{-m})}{(x_{ij} + m\mathbf{a})^2}. \tag{270}$$

In the above, we cannot drop terms with $i = j$ anymore, since they are now spin dependent rather than constant. Only the term $i = j, m = 0$ must be dropped from the summation as before.

Again, the form of the Hamiltonian for the reduced model and its physical interpretation simplifies with an appropriate choice of basis for the spins which makes U diagonal: $U = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi_a})$. The trace in (270) then becomes

$$\mathrm{tr}(S_i U^m S_j U^{-m}) = \sum_{a,b=1}^{n} (S_i)_{ab} (S_j)_{ba} \, \mathrm{e}^{-\mathrm{i}m\phi_{ab}}, \tag{271}$$

where $\phi_{ab} = \phi_a - \phi_b$. The m-summation appearing in (270) gives

$$\sum_{m=-\infty}^{\infty} \frac{\mathrm{tr}(S_i U^m S_j U^{-m})}{(x_{ij} + m\mathbf{a})^2} = \sum_{a,b=1}^{n} (S_i)_{ab} (S_j)_{ba} \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}m\phi_{ab}}}{(x_{ij} + m\mathbf{a})^2} \tag{272}$$

$$= \sum_{a,b=1}^{n} V_{ab}(x_{ij}) (S_i)_{ab} (S_j)_{ba}, \tag{273}$$

with the potential $V_{ab}(x)$ being

$$V_{ab}(x) = \sum_{m=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}m\phi_{ab}}}{(x + m\mathbf{a})^2}. \tag{274}$$

We must distinguish between the cases $i \neq j$ and $i = j$. For the case $i \neq j$, the sum can be obtained from the x-derivative of (257):

$$V_{ab}(x) = \frac{1}{\mathbf{a}^2} \mathrm{e}^{-\mathrm{i}\frac{x}{\mathbf{a}}\phi_{ab}} \left(\frac{\pi^2}{\sin^2 \frac{\pi x}{\mathbf{a}}} - \mathrm{i}\pi\phi_{ab} \cot \frac{\pi x}{\mathbf{a}} - \pi |\phi_{ab}| \right). \tag{275}$$

For the case $i = j$, we must omit the term $m = 0$ from the summation. We obtain an x-independent potential

$$\tilde{V}_{ab} \equiv \lim_{x \to 0} \left(V_{ab}(x) - \frac{1}{x^2} \right) = \frac{\phi_{ab}^2}{2\mathbf{a}^2} - \frac{\pi |\phi_{ab}|}{\mathbf{a}^2}. \tag{276}$$

In the above, we omitted a constant (a, b-independent) term equal to $\frac{\pi^2}{3\mathbf{a}^2}$ which would contribute to the Hamiltonian a term proportional to $\sum_i \mathrm{tr}(S_i)^2$. Due to (262), this is an irrelevant constant. With the above, the reduced Hamiltonian eventually becomes

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \sum_{a,b} V_{ab}(x_{ij})(S_i)_{ab}(S_j)_{ba} + \frac{1}{2} \sum_i \sum_{a,b} \tilde{V}_{ab}(S_i)_{ab}(S_i)_{ba}. \tag{277}$$

This is a model of particles with $U(n)$ spins interacting through $U(n)$ *non*-invariant couplings, due to the presence of the matrix V_{ab}. The original global $U(n)$-invariance is, now, broken to the diagonal $U(1)^n$ part and only the diagonal components S_{aa} of the total spin

$$S_{ab} = \sum_i (S_i)_{ab} \tag{278}$$

are conserved. The standard $U(n)$-invariant spin-Sutherland model is recovered upon choosing $V_{ab} \sim \delta_{ab}$, in which case the sums over a, b above become a normal trace. This is achieved by choosing $\phi_a = \text{constant}$, that is, $U = \mathrm{e}^{\mathrm{i}\phi}$.

The above model is, in fact, the same as the classical model introduced by Blom and Langmann [38] and the author of this paper [40], in the particle–spin form in which it was recast in [40]:

$$\begin{aligned} H = {} & \frac{1}{2} \sum_i p_i^2 + \frac{1}{2} \sum_{i \neq j} \left(\sum_{ij} V_{ab}(x_{ij})(\hat{S}_i)_{ab}(\hat{S}_j)_{ba} + \frac{\ell(\ell+n)}{4n \sin^2 \frac{x_{ij}}{2}} \right) \\ & + \frac{1}{2} \sum_i \sum_{ab} \tilde{V}_{ab}(\hat{S}_i)_{aa}(\hat{S}_i)_{bb} + \frac{1}{2N} \sum_{ab} \tilde{V}_{ab}(q_a q_b - \hat{S}_{aa}\hat{S}_{bb}). \end{aligned} \tag{279}$$

To fully see the equivalence, we must observe the following:

(1) In the present construction, we expressed the Hamiltonian in terms of $U(n)$ spins S_i. In [40], it was, instead, expressed in terms of traceless $SU(n)$ spins $\hat{S}_i$. By (264) and (266) we have $\mathrm{tr}\, S_i = \ell$, so the relation between the two is

$$\hat{S}_i = S_i - \frac{\ell}{n}. \tag{280}$$

(2) Expression (279) derived in [40] was fully quantum mechanical. It can be seen that the term $\ell(\ell+n)$ in (279) classically becomes ℓ^2 (n was a quantum correction similar to the shift of the classical angular momentum J^2 to $J(J+1)$).

(3) For the rank-1 matrices S_i, we have the relation

$$(S_i)_{ab}(S_i)_{ba} = (S_i)_{aa}(S_i)_{bb}. \tag{281}$$

(4) In [40], a set of dynamically conserved charges q_a was introduced that can be chosen to have any value as long as they sum to zero.

(5) In [40], the particles were taken to move on the unit circle, that is, $\mathbf{a} = 2\pi$.

Doing the above substitutions in (277), we see that it becomes practically identical to (279). The two expressions differ by constant terms depending on the charges q_a and the diagonal elements of the total spin S_{aa}. Since both of these quantities are constants of the motion, the two models are trivially related.

(g) We can, similarly to (c), extend the above construction to two complex translations and corresponding spin rotations. Parametrizing again the infinite number of particles with a doublet of indices $i, j \in \mathbf{Z}$, the constraints are

$$\begin{aligned} & x_{i,j} = x_{i+N',j} + \mathbf{a}, \qquad p_{i,j} = p_{i+N',j}, \qquad S_{i+N',j} = U S_{ij} U^{-i}, \\ & x_{i,j} = x_{i,j+M'} + \mathbf{b}, \qquad p_{i,j} = p_{i,j+M'}, \qquad S_{i,j+M'} = V S_{i,j} V^{-1} \end{aligned} \tag{282}$$

and we end up as before with a finite system with $N = M'N'$ particles periodically repeating on the complex plane. Choosing $M' = 1$, $N' = N$, the spin matrices $S_{i+mN,n} \equiv S_{i;m,n}$ are now expressed as

$$S_{i;m,n} = U^m V^n S_i V^{-n} U^{-m} = V^n U^m S_i U^{-m} V^{-n}. \tag{283}$$

The corresponding reduced Hamiltonian becomes infinite copies of

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i,j} \sum_{m,n=-\infty}^{\infty} \frac{\mathrm{tr}(U^m V^n S_i V^{-n} U^{-m} S_j)}{(x_{ij} + m\mathbf{a} + n\mathbf{b})^2}. \tag{284}$$

Since the two space translations and the corresponding particle permutations commute, for consistency the two spin rotations must also commute:

$$UV S_{i;m,n} V^{-1} U^{-1} = VU S_{i;m,n} U^{-1} V^{-1}, \tag{285}$$

which implies

$$[U^{-1} V^{-1} U V, S_{i;m,n}] = 0. \tag{286}$$

For this to hold for generic S_i, we must require $U^{-1} V^{-1} U V \equiv \omega$ to be proportional to the identity matrix. Clearly, ω satisfies $\det(\omega) = \omega^n = 1$, so we obtain

$$UV = \omega VU, \qquad \omega = \mathrm{e}^{\mathrm{i}2\pi \frac{\nu}{n}}, \tag{287}$$

with ν an integer $0 \leqslant \nu < n$. U and V then satisfy Weyl's braiding condition which characterizes a noncommutative ('quantum') torus [63].

For $\nu = 0$ ($\omega = 1$), the matrices U and V commute. For $\omega \neq 1$, however, U and V become 'clock' and 'shift' matrices. We shall deal with the two cases separately.

7.4. Twisted elliptic spin-Calogero models

In this case, U and V can be simultaneously diagonalized. Just as in the previous case, we choose a basis for the spins that diagonalizes both U and V to $U = \mathrm{diag}(\phi_i)$, $V = \mathrm{diag}(\theta_i)$. The m, n-sums that appear in (284) become

$$\sum_{a,b=1}^{n} (S_i)_{ab} (S_j)_{ba} \sum_{m,n=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}m\phi_{ab} - \mathrm{i}n\theta_{ab}}}{(x_{ij} + m\mathbf{a} + n\mathbf{b})^2}, \tag{288}$$

where the term $m = n = 0$ is omitted if $i = j$. We again obtain a potential $V_{ab}(x_{ij})$, for $i \neq j$, given by the sum

$$V_{ab}(x) = \sum_{m,n=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}m\phi_{ab} - \mathrm{i}n\theta_{ab}}}{(x + m\mathbf{a} + n\mathbf{b})^2}, \tag{289}$$

and a spin self-coupling $\tilde{V}_{ab}$ for $i = j$, given by

$$\tilde{V}_{ab} = \sum_{(m,n) \neq (0,0)} \frac{\mathrm{e}^{-\mathrm{i}m\phi_{ab} - \mathrm{i}n\theta_{ab}}}{(m\mathbf{a} + n\mathbf{b})^2}. \tag{290}$$

Note that now, due to the presence of the phase factors, these sums are convergent and have no regularization ambiguity. The only ambiguous terms, defined modulo an additive constant, are those with $a = b$. We will comment on the impact of such a regularization ambiguity in the following.

The potential $V_{ab}(x)$ is a modular function on the complex torus $(\mathbf{a}, \mathbf{b})$ with quasiperiodicity

$$V_{ab}(x + \mathbf{a}) = \mathrm{e}^{\mathrm{i}\phi_{ab}} V_{ab}(x), \qquad V_{ab}(x + \mathbf{b}) = \mathrm{e}^{\mathrm{i}\theta_{ab}} V_{ab}(x). \tag{291}$$

It has a double pole at $x = 0$, with principal part

$$V_{ab}(x) = \frac{1}{x^2} + O(x^0) \tag{292}$$

and no other poles in each cell. These properties uniquely define V_{ab} and allow for an expression in terms of theta functions. We put

$$V_{ab}(x) = A\, \mathrm{e}^{\mathrm{i}\frac{x}{\mathbf{a}}\phi_{ab}} \frac{\theta_1\left(\frac{\pi}{\mathbf{a}}(x-q_1)\right)\theta_1\left(\frac{\pi}{\mathbf{a}}(x-q_2)\right)}{\theta_1\left(\frac{\pi}{\mathbf{a}}x\right)^2}, \tag{293}$$

where $q_{1,2}$ are the as yet unknown zeros of $V_{ab}(x)$ and the theta functions appearing above have complex period $T = \mathbf{b}/\mathbf{a}$. This has the right quasiperiodicity under $x \to x + \mathbf{a}$. In order to also have the right quasiperiodicity under $x \to x + \mathbf{b}$, $q_{1,2}$ must satisfy

$$q_1 + q_2 = \frac{1}{2\pi}(\mathbf{a}\theta_{ab} - \mathbf{b}\phi_{ab}) \equiv Q_{ab} \tag{294}$$

and to have the right behaviour around $x = 0$ we must further have

$$A = \frac{\pi^2\theta_1'(0)^2}{\mathbf{a}^2\theta_1\left(\frac{\pi q_1}{\mathbf{a}}\right)\theta_1\left(\frac{\pi q_2}{\mathbf{a}}\right)}, \tag{295}$$

$$\frac{\theta_1'\left(\frac{\pi}{\mathbf{a}}q_1\right)}{\theta_1\left(\frac{\pi}{\mathbf{a}}q_1\right)} + \frac{\theta_1'\left(\frac{\pi}{\mathbf{a}}q_2\right)}{\theta_1\left(\frac{\pi}{\mathbf{a}}q_2\right)} = \mathrm{i}\frac{\phi_{ab}}{\pi}. \tag{296}$$

Equations (294) and (296) determine q_1 and q_2, while (295) then determines A. It may be possible to express q_1, q_2 in a more explicit form or to recast (293) in a form more symmetric in $\mathbf{a}$, $\mathbf{b}$, by using theta-function identities. Finally, the self-coupling $\tilde{V}_{ab}$ can be extracted from $V_{ab}(x)$ as

$$\tilde{V}_{ab} = \lim_{x\to 0}\left(V_{ab}(x) - \frac{1}{x^2}\right). \tag{297}$$

To sum up, we obtain a $U(n)$ non-invariant spin generalization of the elliptic model given by a Hamiltonian of the form (277) but with the potentials appearing now being given by (293), (297). The $U(n)$-invariance of the original model is, again, broken down to the diagonal Abelian subgroup $U(1)^n$ due to the dependence of the potential on a, b. The $U(n)$-invariant spin-Weierstrass model is regained for $\phi_{ab} = \theta_{ab} = 0$, that is, trivial matrices U and V.

We point out that for $\theta_{ab} = \phi_{ab} = 0$, that is, $Q_{ab} = 0$, the equations for $q_{1,2}$ ((294), (296)) are satisfied for *any* $q_1 = -q_2$ leading to an apparent arbitrariness. As can be seen, however, by applying the addition formula

$$\theta_1(x+q)\theta_1(x-q)\theta_4(0)^2 = \theta_1(x)^2\theta_4(q)^2 - \theta_4(x)^2\theta_1(q)^2, \tag{298}$$

this simply amounts to an arbitrary additive constant to the expression for $V_{ab}(x) \equiv V(x)$. This corresponds to the need for regularization for this expression in the absence of phases, as explained before. (In the case of the Weierstrass function, this is fixed by further requiring that the $O(x^0)$ part of the function at $x = 0$ vanish, which picks $q = \pi T/2$ and makes $\theta_4(q)$ above vanish.) We also point out that we can pick *any* of these values for $q_1 = -q_2$ at the limit $Q = 0$ by appropriately choosing the ratio ϕ_{ab}/θ_{ab} as they both go to zero.

The ambiguity of the terms with $a = b$ can be fixed in the same way: we can choose phases $\phi_{aa} \neq 0$, $\theta_{aa} \neq 0$, evaluate the expressions and then let $\phi_{aa}, \theta_{aa} \to 0$. This will lead to arbitrary additive constants C_a, depending on the ratio ϕ_{aa}/θ_{aa} as we take them to zero. The

same constants, however, will appear in both $V_{aa}(x)$ and $\tilde{V}_{aa}$. Their net contribution to the Hamiltonian will be

$$\Delta H = \frac{1}{2}\sum_{i\neq j}\sum_a C_a (S_i)_{aa}(S_j)_{aa} + \frac{1}{2}\sum_i\sum_a C_a (S_i)_{aa}(S_i)_{aa} = \frac{1}{2}\sum_a C_a (S_{aa})^2. \tag{299}$$

Since the diagonal components of the total spin S are still constants of the motion, due to the residual $U(1)^n$-invariance, this amounts to the addition of an overall constant, and thus leads to systems that are trivially related.

The same discussion applies if the angles ϕ_a and θ_a coincide for two or more values of a belonging to a subspace of indices I, in which case $\phi_{ab} = \theta_{ab} = 0$ for $a, b \in I$. This will result in a constant additive matrix C_{ab} in the potential for this subspace of indices I, leading to an extra contribution to the Hamiltonian

$$\Delta H = \sum_{a,b\in I} C_{ab} S_{ab} S_{ba}. \tag{300}$$

Since $\phi_{ab} = \theta_{ab} = 0$ for $a, b \in I$, however, the corresponding subgroup of $U(n)$ remains unbroken, and thus the corresponding components of the total spin S_{ab} appearing above are constants of the motion. Once again, the arbitrary terms are constant and we essentially obtain a unique system.

Overall, this is a generalization of the spin-Weierstrass model to one involving $2n$ phases that break the $U(n)$-invariance and promote the potential to a modular function. The potential lives on a complex torus in the coordinates, where translations around each nontrivial cycle are accompanied by spin transformations. The model obtained in (f) can be though of as the limit of the present model with $\mathbf{b} \to \infty$, in which case θ_a become irrelevant.

7.5. Noncommutative spin-Calogero model

In this case,

$$UV = \omega VU, \qquad \omega = \mathrm{e}^{\mathrm{i}2\pi\frac{\nu}{n}}, \tag{301}$$

with ν an integer $0 < \nu < n$.

To proceed, we must identify the form of U, V. We need the irreducible representations of relation (301). Call k the greatest common divisor of ν and n. Then $n = km$ and $\nu = k\mu$, for relatively prime m, μ. The irreducible representations for U, V are m-dimensional 'clock' and 'shift' matrices. By a global $U(n)$ spin transformation we can diagonalize either of U, V. Choosing U diagonal, the general form of U and V will be the direct sum of k of the above irreducible representations:

$$U = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}\phi_0}, \ldots, \mathrm{e}^{\mathrm{i}\phi_{k-1}}\} \otimes u, \qquad V = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}\theta_0}, \ldots, \mathrm{e}^{\mathrm{i}\theta_{k-1}}\} \otimes v, \tag{302}$$

where ϕ_q, θ_q are arbitrary phases, determining the Casimirs U^m and V^m, and u, v are the m-dimensional clock and shift matrices

$$u_{\alpha\beta} = \omega^\alpha \delta_{\alpha\beta}, \qquad v_{\alpha\beta} = \delta_{\alpha+1,\beta} \ (\mathrm{mod}\ m), \qquad \alpha, \beta = 0, \ldots, m-1. \tag{303}$$

So the acceptable U and V depend on $2k$ arbitrary parameters.

To take advantage of the form (302) for U, V we partition S_i into k^2 blocks of dimension $m \times m$ each by using the double index notation

$$(S_i)_{ab} = (S_i)^{pq}_{\alpha\beta}, \qquad a = pm + \alpha, \quad b = qm + \beta. \tag{304}$$

The $U(n)$ Poisson brackets in this notation are

$$\left\{(S_i)^{pq}_{\alpha\beta}, (S_j)^{rs}_{\gamma\delta}\right\} = -\mathrm{i}\delta_{ij}\left[(S_i)^{ps}_{\alpha\delta}\delta_{\gamma\beta}\delta_{rq} - \delta_{\alpha\delta}\delta_{rq}(S_i)^{rq}_{\gamma\beta}\right]. \tag{305}$$

The m, n-sums that appear in (284) then become

$$\sum_{m,n;\alpha,\beta;\,p,q} (S_i)^{pq}_{\alpha+n,\beta+n}(S_j)^{qp}_{\beta,\alpha}\frac{\mathrm{e}^{-\mathrm{i}m\phi_{pq}-\mathrm{i}n\theta_{pq}}\omega^{m(\alpha-\beta)}}{(x_{ij}+m\mathbf{a}+n\mathbf{b})^2}, \tag{306}$$

where the term $m = n = 0$ is omitted if $i = j$.

The above gives a potential interaction between particles i and j in the form of a modular function in x_{ij} which depends on the spin components of particles i and j. To make the noncommutative character of the spin interaction manifest, we perform a change of basis in the spin states and define

$$(\tilde{S}_i)^{pq}_{\alpha\beta} = \sum_{\sigma}\omega^{(\sigma+\frac{\alpha}{2})\beta}(S_i)^{pq}_{\alpha+\sigma,\sigma}. \tag{307}$$

This is essentially a discrete Fourier transform in the sum of the α, β indices of $S^{pq}_{\alpha\beta}$. (Note that, for m odd, $\tilde{S}^{pq}_{\alpha\beta}$ is actually antiperiodic in the index α if β is odd, and vice versa. Although we could have defined a properly periodic matrix, we prefer this slight inconvenience in order to make the ensuing formulae more symmetric.) In fact, it will be convenient to assemble the double indices (α, β) and (m, n) into vectors $\vec{\alpha}$ and $\vec{m}$. Similarly, we define $\vec{\mathbf{c}} = (\mathbf{a}, \mathbf{b})$ and $\vec{\phi}_p = (\phi_p, \theta_p)$.

The Poisson brackets of $\tilde{S}_i$ are found from (305)

$$\big\{(\tilde{S}_i)^{pq}_{\vec{\alpha}}, (\tilde{S}_j)^{rs}_{\vec{\beta}}\big\} = \mathrm{i}\delta_{ij}\big[\omega^{\frac{\vec{\alpha}\times\vec{\beta}}{2}}\delta_{ps}(\tilde{S}_i)^{rq}_{\vec{\alpha}+\vec{\beta}} - \omega^{-\frac{\vec{\alpha}\times\vec{\beta}}{2}}\delta_{rq}(\tilde{S}_i)^{ps}_{\vec{\alpha}+\vec{\beta}}\big]. \tag{308}$$

This is a structure extending the Moyal (star-commutator) algebra, the exponent of ω being the cross product of the discrete 'momenta' $\vec{\alpha}$ and $\vec{\beta}$. For $(rs) = (pq)$, in particular, it becomes the torus Fourier transform of the Moyal bracket

$$\big\{(\tilde{S}_i)^{pq}_{\vec{\alpha}}, (\tilde{S}_j)^{pq}_{\vec{\beta}}\big\} = \mathrm{i}\delta_{ij}\big(\omega^{\frac{1}{2}} - \omega^{-\frac{1}{2}}\big)[\vec{\alpha}\times\vec{\beta}]_\omega(\tilde{S}_i)^{pq}_{\vec{\alpha}+\vec{\beta}}, \tag{309}$$

where

$$[x]_\omega = \frac{\omega^{\frac{x}{2}} - \omega^{-\frac{x}{2}}}{\omega^{\frac{1}{2}} - \omega^{-\frac{1}{2}}} \tag{310}$$

is the ω-deformation of x. This is the so-called trigonometric algebra with periodic discrete indices [15].

Finally, by inverting (307) and substituting in (306), the potential energy W in terms of $\tilde{S}_i$ acquires the form

$$W = \sum_{i,j}\sum_{\vec{\alpha};\,p,q}(\tilde{S}_i)^{pq}_{\vec{\alpha}}(\tilde{S}_j)^{qp}_{-\vec{\alpha}}W^{pq}_{\vec{\alpha}}(x_{ij}). \tag{311}$$

The above includes two-body interactions, for $i \neq j$, as well as spin self-couplings, for $i = j$, arising from the interaction of each particle with its own images in different cells. The two-body potential $W^{pq}_{\vec{\alpha}}(x)$ is

$$W^{pq}_{\vec{\alpha}}(x) = \frac{1}{m}\sum_{\vec{m}}\frac{\omega^{\vec{\alpha}\times\vec{m}}\,\mathrm{e}^{\mathrm{i}\vec{\phi}_{pq}\cdot\vec{m}}}{(x+\vec{\mathbf{c}}\cdot\vec{m})^2}, \tag{312}$$

while the spin self-coupling $\tilde{W}^{pq}_{\vec{\alpha}}$ is

$$\tilde{W}^{pq}_{\vec{\alpha}} = \frac{1}{m}\sum_{\vec{m}\neq\vec{0}}\frac{\omega^{\vec{\alpha}\times\vec{m}}\,\mathrm{e}^{\mathrm{i}\vec{\phi}_{pq}\cdot\vec{m}}}{(\vec{\mathbf{c}}\cdot\vec{m})^2}. \tag{313}$$

If the above potentials were independent of the $U(n)$ indices $\vec{\alpha}$ and p, q, the sum over $U(n)$ indices in the potential energy expression (311) would simply be a $U(n)$ trace and

would give the $U(n)$-invariant coupling between the spins of particles i and j multiplying the standard Weierstrass potential of the elliptic Calogero model. In the present case, however, the above potential is spin dependent and breaks $U(n)$-invariance, introducing a star-product twist in the indices $\vec{\alpha}$ and phase shifts $\vec{\phi}_p$ in the indices p, q. Generically, the $U(n)$-invariance of the original model is broken down to an Abelian $U(1)^k$, amounting to the transformation

$$(S_i)^{pq}_{\alpha\beta} \to \mathrm{e}^{\mathrm{i}\varphi_p}(S_i)^{pq}_{\alpha\beta}\,\mathrm{e}^{-\mathrm{i}\varphi_q}. \tag{314}$$

If $\vec{\phi}_p$ are equal for k' values of p, the remaining symmetry $U(1)^{k'}$ is enhanced to $U(k')$, corresponding to mixing the corresponding p-components.

The case $\omega = 1, \vec{\phi}_p = 0$ reduces to the standard spin-elliptic Calogero–Moser model. The case $m = 1$ (and thus $\omega = 1$) reproduces the $U(n)$ non-invariant model of the previous subsection. The general case with $\omega \neq 1$ is a new classical integrable model of the spin-Calogero type with a spin-dependent potential which is a modular function of the two-body distance x_{ij}.

The sums appearing in (312) and (313) could in principle have ambiguities due to the logarithmic divergence of the summation over the radial coordinate on the complex plane. This is, indeed, the case for the standard Weierstrass function and a specific prescription is needed to regularize it. Different prescriptions lead to different additive constants in the result. In our case, however, the presence of the extra phases renders the sums convergent and there is no regularization ambiguity.

The potentials can be expressed in terms of theta functions. $W^{pq}_{\vec{\alpha}}(x)$ is a modular function on the complex torus $(\mathbf{a}, \mathbf{b})$ with quasiperiodicity

$$\begin{aligned} W^{pq}_{\vec{\alpha}}(x+\mathbf{a}) &= \exp\left(-\mathrm{i}\phi_{pq} + \frac{2\pi\mu}{m}\alpha_2\right) W^{pq}_{\vec{\alpha}}(x), \\ W^{pq}_{\vec{\alpha}}(x+\mathbf{b}) &= \exp\left(-\mathrm{i}\theta_{pq} - \frac{2\pi\mu}{m}\alpha_1\right) W^{pq}_{\vec{\alpha}}(x). \end{aligned} \tag{315}$$

It has a double pole at $x = 0$, with principal part

$$W^{pq}_{\vec{\alpha}}(x) = \frac{1}{mx^2} + O(x^0) \tag{316}$$

and no other poles in each cell. These properties uniquely define $W^{pq}_{\vec{\alpha}}(x)$ and allow for an expression in terms of theta functions. We put

$$W^{pq}_{\vec{\alpha}}(x) = A\omega^{-\mathrm{i}\frac{x}{\mathbf{a}}}\,\mathrm{e}^{-\mathrm{i}\frac{x}{\mathbf{a}}\phi_{pq}}\frac{\theta_1\left(\frac{\pi}{\mathbf{a}}(x-Q_1)\right)\theta_1\left(\frac{\pi}{\mathbf{a}}(x-Q_2)\right)}{\theta_1\left(\frac{\pi}{\mathbf{a}}x\right)^2}, \tag{317}$$

where $Q_{1,2}$ are the as yet unknown zeros of $W^{pq}_{\vec{\alpha}}(x)$ and the theta functions appearing above have complex period $T = \mathbf{b}/\mathbf{a}$. This has the right quasiperiodicity under $x \to x + \mathbf{a}$. In order to also have the right quasiperiodicity under $x \to x + \mathbf{b}$, $Q_{1,2}$ must satisfy

$$Q_1 + Q_2 = \frac{\vec{\phi}_{ab} \times \vec{\mathbf{c}}}{2\pi} + \frac{\mu}{m}\vec{\alpha}\cdot\vec{\mathbf{c}} \tag{318}$$

and to have the right behaviour around $x = 0$ we must further have

$$\frac{\theta_1'\left(\frac{\pi}{\mathbf{a}}Q_1\right)}{\theta_1\left(\frac{\pi}{\mathbf{a}}Q_1\right)} + \frac{\theta_1'\left(\frac{\pi}{\mathbf{a}}Q_2\right)}{\theta_1\left(\frac{\pi}{\mathbf{a}}Q_2\right)} = -\mathrm{i}\frac{\phi_{ab}}{\pi} - 2\mathrm{i}\frac{\mu}{m}\alpha_1, \tag{319}$$

$$A = \frac{\pi^2\theta_1'(0)^2}{m\mathbf{a}^2\theta_1\left(\frac{\pi Q_1}{\mathbf{a}}\right)\theta_1\left(\frac{\pi Q_2}{\mathbf{a}}\right)}. \tag{320}$$

Equations (318) and (319) determine Q_1 and Q_2, while (320) in turn determines A. The self-coupling $\tilde{W}_{\vec{\alpha}}^{pq}$ can then be extracted from $W_{\vec{\alpha}}^{pq}(x)$ as

$$\tilde{W}_{\vec{\alpha}}^{pq} = \lim_{x\to 0}\left(W_{\vec{\alpha}}^{pq}(x) - \frac{1}{mx^2}\right). \tag{321}$$

The sums appearing in (312) and (313) are in general convergent, due to the presence of the phases. For $\omega = 1$, however, the phases are absent and terms with $p = q$ have an additive ambiguity due to the need for regularization for expression (312). In the theta-function expression, this manifests in the fact that the equations for $Q_{1,2}$ ((318), (319)) are satisfied for *any* $Q_1 = -Q_2$. By applying the addition formula

$$\theta_1(x+Q)\theta_1(x-Q)\theta_4(0)^2 = \theta_1(x)^2\theta_4(Q)^2 - \theta_4(x)^2\theta_1(Q)^2, \tag{322}$$

this is seen indeed to amount to an arbitrary additive constant to the expression for $W^{pp}(x)$. The same holds for terms p, q for which $\vec{\phi}_{pq} = 0$. Such arbitrariness, however, corresponds to trivial redefinitions of the model by addition of constants of motion, as explained in the previous subsection.

In conclusion, we identified an integrable generalization of the elliptic spin model which breaks the spin $U(n)$-invariance and promotes the potential to a modular function introducing noncommutative spin twists.

8. Epilogue

This concludes our promenade in Calogero land. We have seen and touched many aspects of these models, but have by no means exhausted them. There are various other issues that have not been visited. A few of these are listed below.

There are systems with nearest-neighbour interactions that can also be obtained and solved in the exchange operator formalism [64]. Such systems were not analysed here. We also mention the alternative operator approaches to analyse such models used in [65].

Spin models that can be solved with the techniques outlined in this paper include many examples not treated here. Apart from the supersymmetric models mentioned earlier, we also list the models treated in [66].

The continuous limit of a system of Calogero particles can be described in the collective field theory formulation [67]. The system exhibits interesting soliton and wave solutions [68, 69], whose chiral nature and connection to Benjamin–Davis–Acrivos–Ono solitons is an interesting issue [70].

Finally, there are certainly lots more topics related to the Calogero model, and related references, to which this paper has not made justice. The most egregious omissions will hopefully be rectified in the next revision.

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